MULTIAXIAL STRESSES

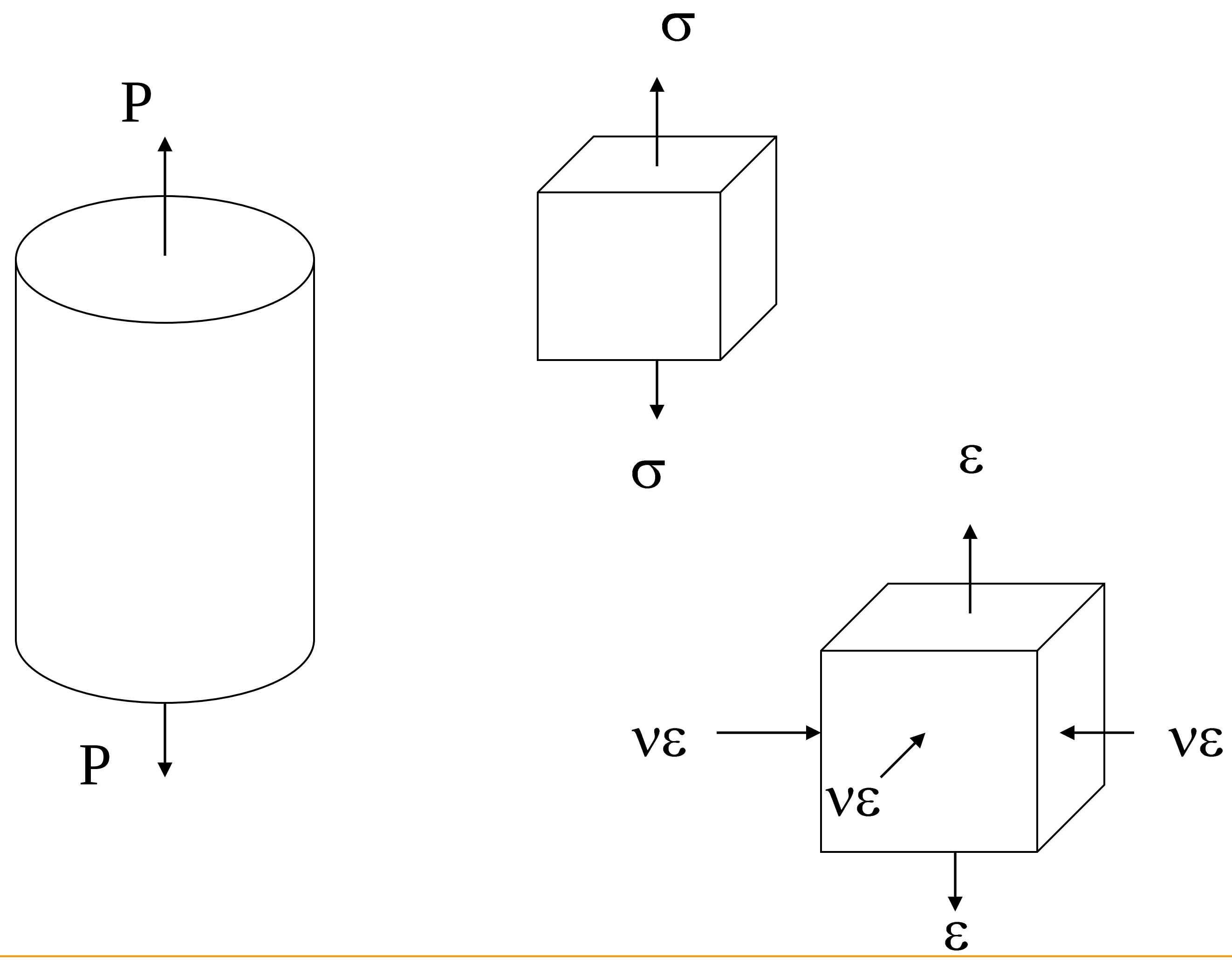
MULTIAXIAL STRESSES

- Multiaxial states of stress are very common and multiaxial strain is difficult to avoid. For **example**:
  - The strains are triaxial, in a tensile bar.
  - In a shaft that transmits torque stress state is biaxial.
  - In a thin-walled pressure vessel subjected to cyclic pressure, the stress state is biaxial.
  - In a crankshaft we have torsion and bending.

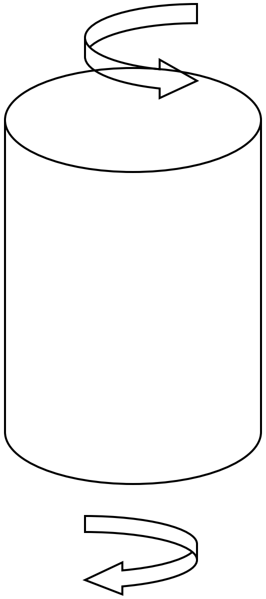
- The state of stress in **notches** is usually multiaxial and not the same as the state of stress in the main body.
  - For example, at the root of a thread the state of stress is biaxial, although it may be uniaxial in the main body of a bolt.

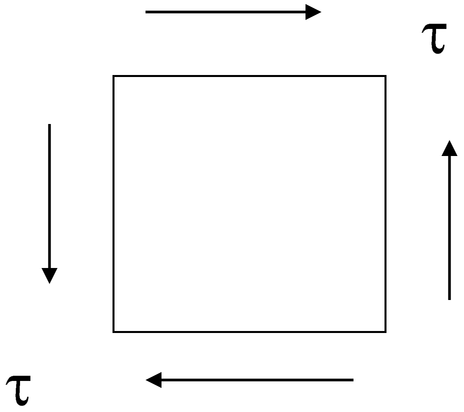
- **Stress concentration factor** changes with stress state.

Uniaxial stress with multiaxial strain

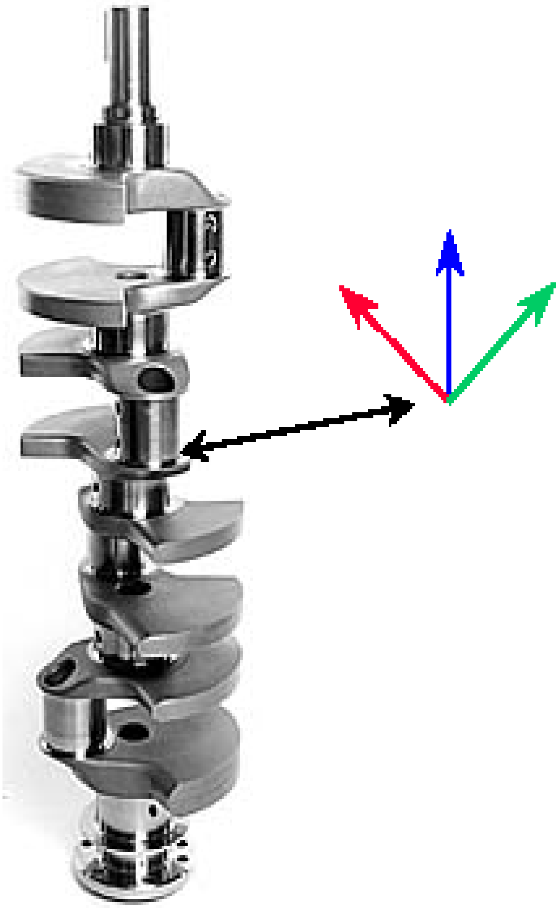


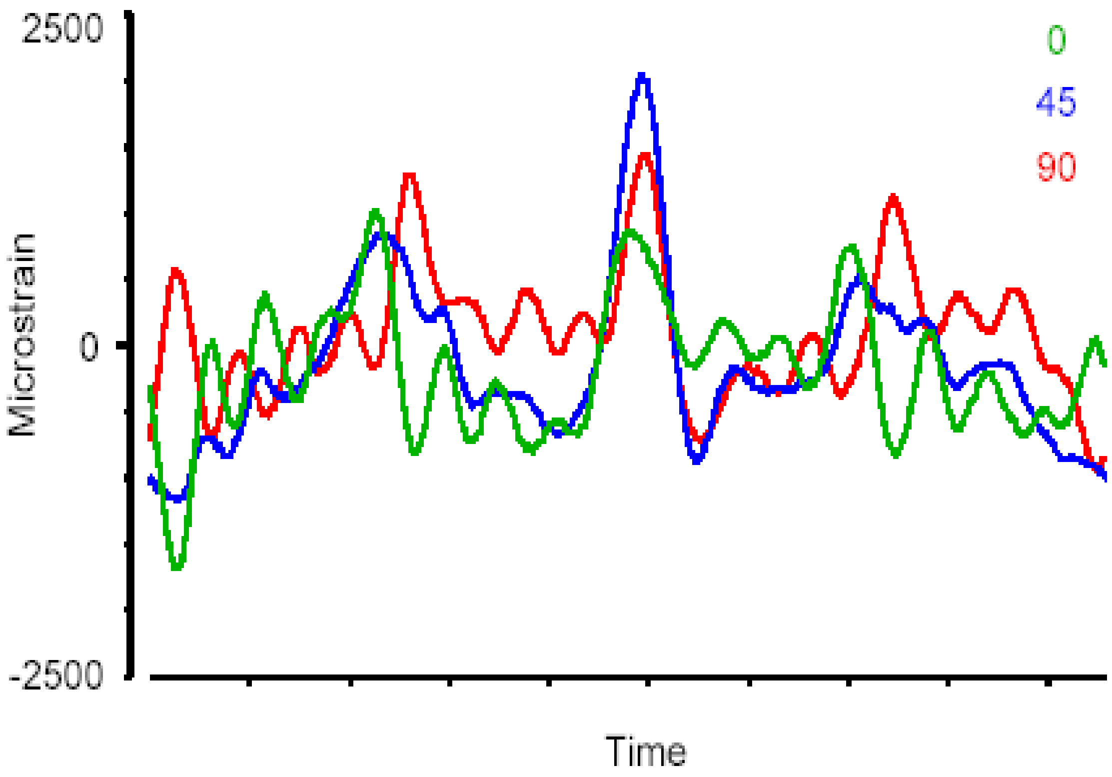
Biaxial stress





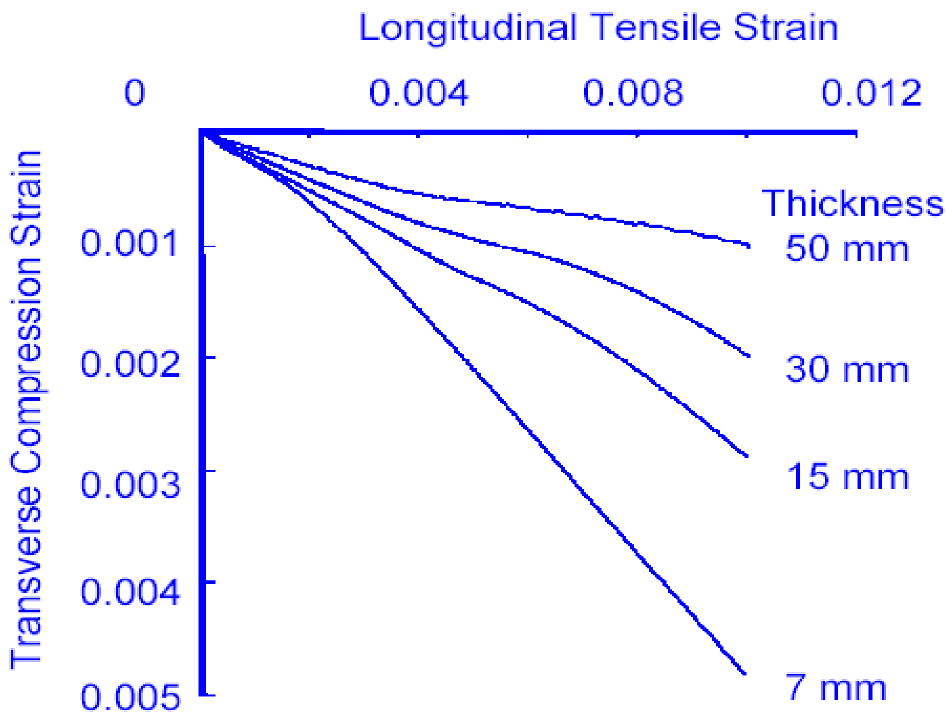
Stresses in a crankshaft
(Courtesy of Darrell Socie, Multiaxail Fatigue)

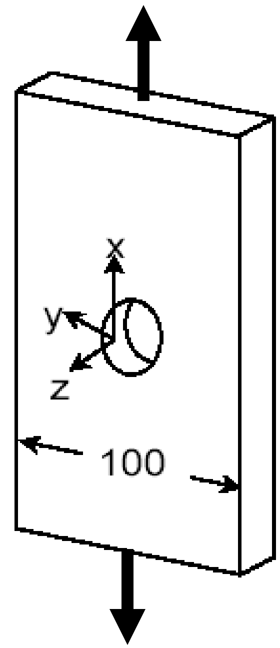




Stress state at a notch in uniaxial tension

(Courtesy of Darrell Socie, Multiaxail Fatigue)





t	ε_x	ε_z	σ_x	σ_z
7	0.01	-0.005	63.5	0
15	0.01	-0.003	70.6	14.1
30	0.01	-0.002	73.0	21.8
50	0.01	-0.001	75.1	29.3

MULTIAXIAL STRESSES

Can we apply our knowledge and data from uniaxial behavior and tests to multiaxial situations? This is the question with which this chapter is concerned.

MULTIAXIAL STRESSES: **CHAPTER OUTLINE**

- Brief review of states of stress and strain
- Classification of constant amplitude multiaxial loading to proportional & nonproportional loading
- Brief discussion of yielding & plasticity for multiaxial stresses.
- Multiaxial fatigue life estimation methods:
  - stress-based approaches,
  - strain-based and energy-based approaches,
  - critical plane models, and
  - fracture mechanics approach for crack growth.
- Brief discussion of notched effects & VA loading.

MULTIAXIAL STRESSES
(STATES OF STRESS AND STRAIN)

- The state of stress and strain at a point in the body can be described by six stress components (σ_x, σ_y, σ_z, τ_{xy}, τ_{xz}, τ_{yz}) and six strain components (ε_x, ε_y, ε_z, γ_{xy}, γ_{xz}, γ_{yz}) acting on orthogonal planes *x*, *y* and *z*.

- Stresses and strains acting in any other direction or plane can be found by using transformation equations or graphically by using Mohr's circle.

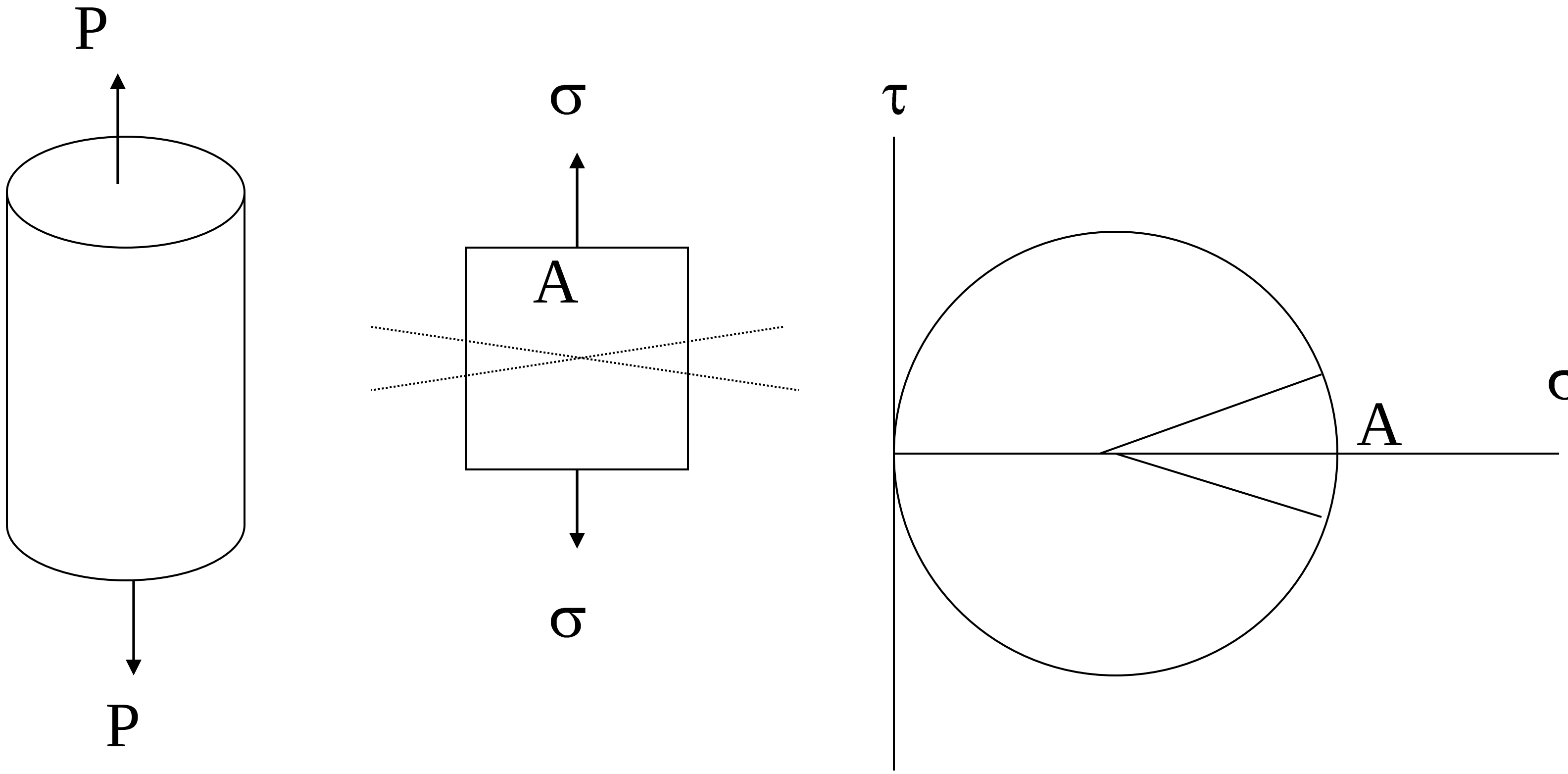
MULTIAXIAL STRESSES
(STATES OF STRESS AND STRAIN)

- Of special interest to fatigue analysis are the magnitudes and directions of the following quantities at a critical location in the component or structure:

  - maximum normal principal stress, σ_1
  - maximum shearing stress, τ_{max}
  - maximum octahedral shearing stress, τ_{oct}
  - maximum normal principal strain, ε_1
  - maximum shearing strain, γ_{max}
  - maximum octahedral shearing strain, γ_{oct}

MULTIAXIAL STRESSES
(STATES OF STRESS AND STRAIN)

- It is important to realize that even though only a few planes experience the maximum principal normal stress (or strain) and the maximum shearing stress (or strain), many other planes can experience a very large percentage of these quantities. For **example**,
  - For the case of simple uniaxial tension even though only the loading plane experiences the maximum normal stress σ, all planes oriented between 13 from the loading plane experience at least 95% of σ.
  - Also, a shearing stress is present on every stressed plane, except for the loading plane.


P
σ
τ
A
σ
A
σ
P

MULTIAXIAL STRESSES
(STATES OF STRESS AND STRAIN)

- The **octahedral planes** are also of importance in yielding prediction and fatigue analysis.
  - There are eight octahedral planes making equal angles with the three principal stress directions.
  - The shearing stress on these planes is given by

$$\tau_{oct} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

  - The normal stress on an octahedral plane is the hydrostatic stress (also called the average normal stress) given by

$$\sigma_{oct} = \sigma_h = \sigma_{ave} = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3)$$

  - The shear strain acting on an octahedral plane is given by

$$\gamma_{oct} = \frac{2}{3}\sqrt{(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2}$$

MULTIAXIAL STRESSES
(PROPORTIONAL VS NONPROPORTIONAL LOADING)

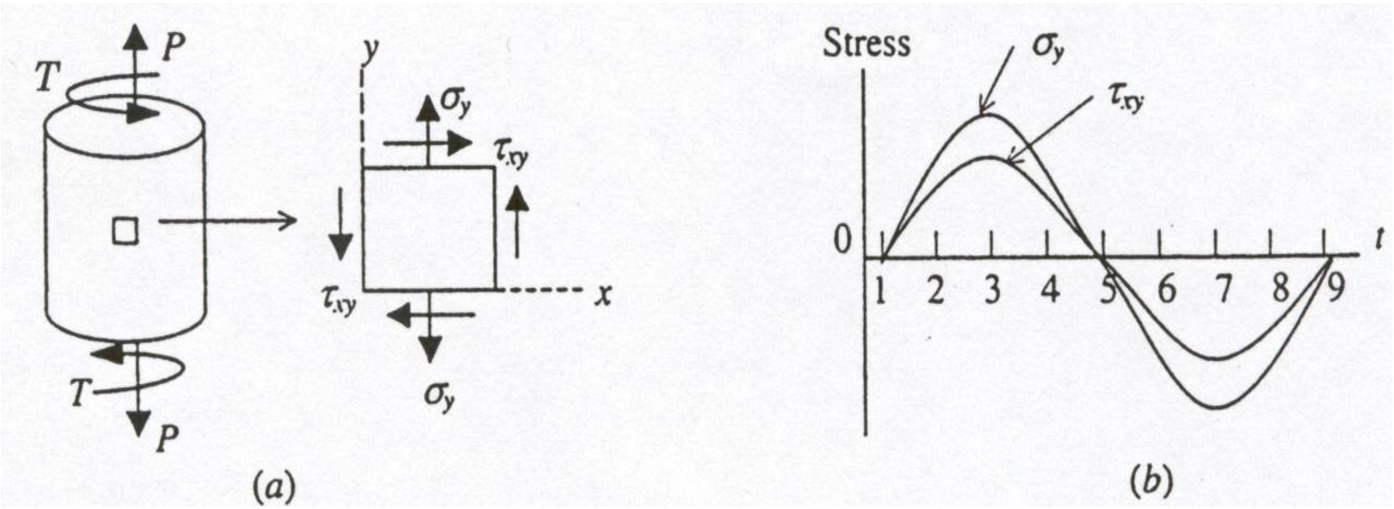
- During constant amplitude cyclic loading, as the magnitude of the applied stresses vary with time, the size of Mohr's circle of stress also varies with time.

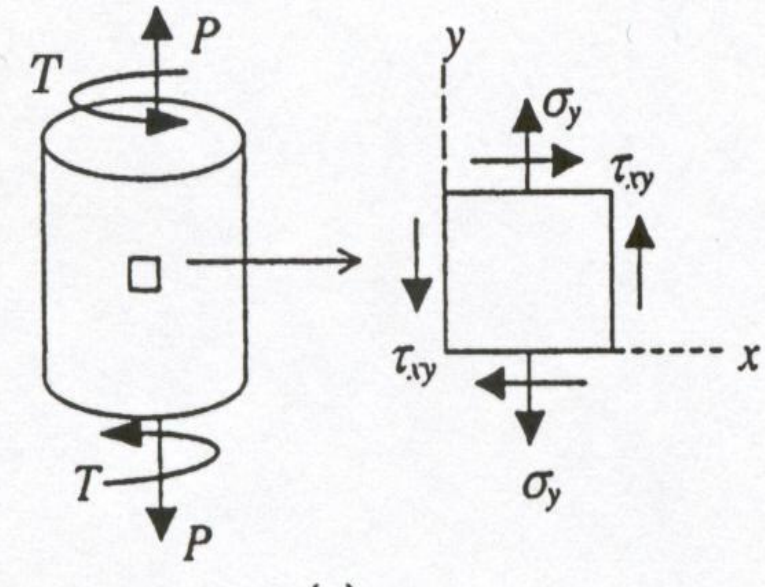
- In some cases, even though the size of the Mohr's circle varies during cyclic loading, the orientation of the principal axes with respect to the loading axes remains fixed. This is called **proportional** loading.

- In many cases, however, the principal directions of the alternating stresses are not fixed, but change orientation. This type of loading is called **nonproportional** loading.
  - Crankshafts are a typical example.
  - Shafts subjected to out-of-phase torsion and bending are another.

MULTIAXIAL STRESSES
(PROPORTIONAL VS NONPROPORTIONAL LOADING)

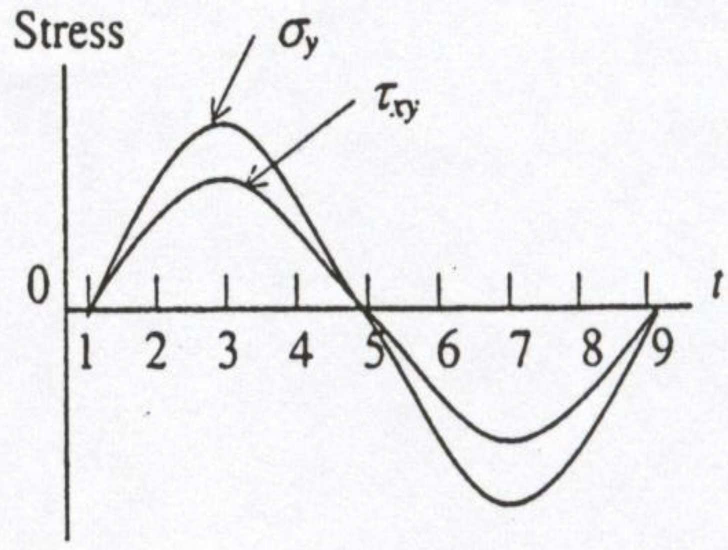
- Proportional and nonproportioinal loading are illustrated for combined axial-torsion loading of a shaft shown in Fig. *a*.
  - The loads in Fig. *b* are applied **in-phase**, so that the maximum and minimum axial and torsion stresses occur simultaneously (Fig. b).
    - This is, therefore, called **proportional** loading.
    - Mohr's circles of stress at times 2 and 3 during the in-phase loading cycle are shown in Fig *e*. (i.e. angle 2α remains constant).





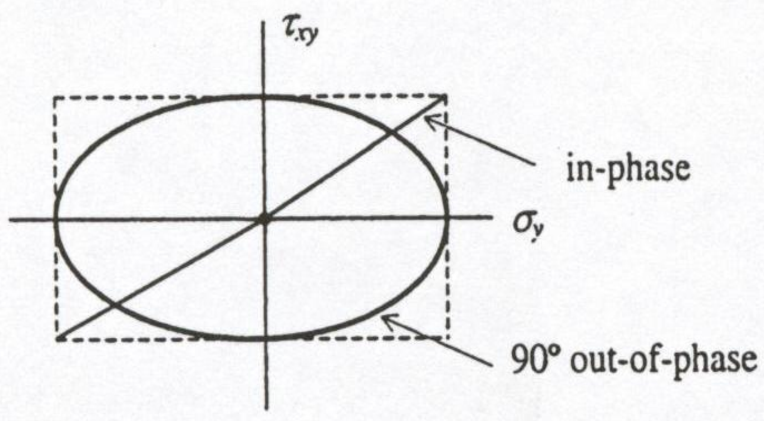
T
P
y
σ_y
τ_{xy}
τ_{xy}
x
σ_y
T
P

(a)



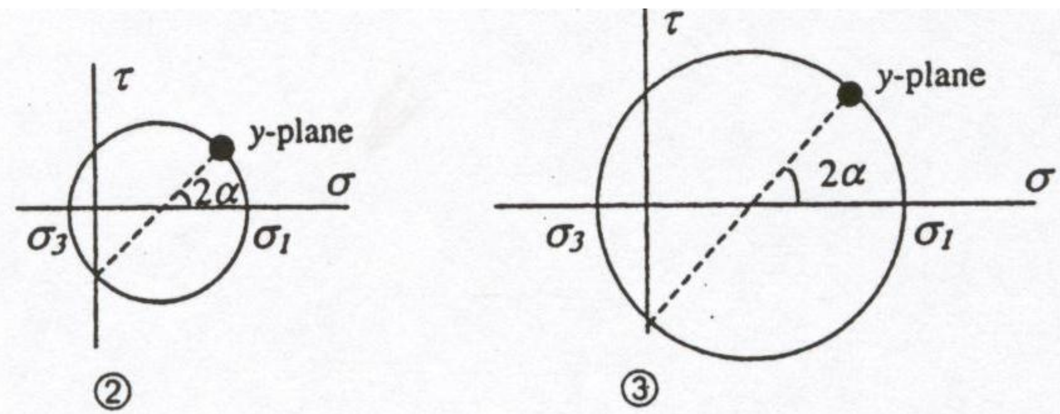
Stress
σ_y
τ_{xy}
0
t
1 2 3 4 5 6 7 8 9

(b)



τ_{xy}
σ_y
in-phase
90° out-of-phase

(c)

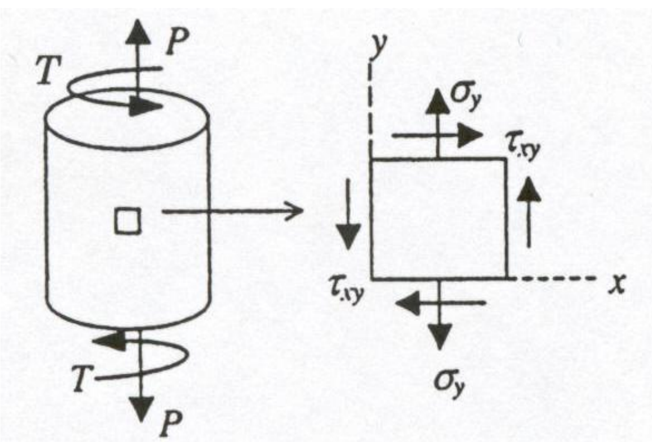


τ
y-plane
σ
2α
σ_3
σ_1
②
τ
y-plane
2α
σ
σ_3
σ_1
③

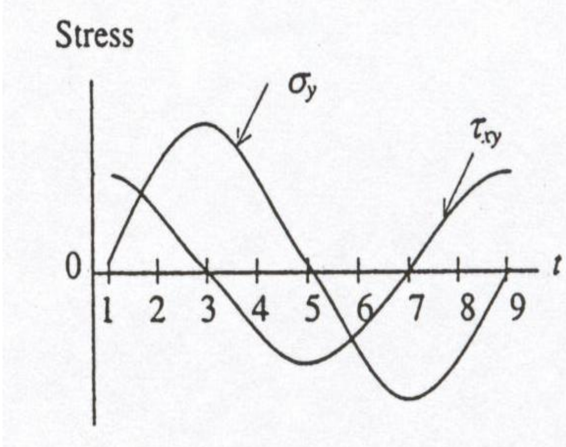
(e)

MULTIAXIAL STRESSES
(**PROPORTIONAL VS NONPROPORTIONAL** LOADING)

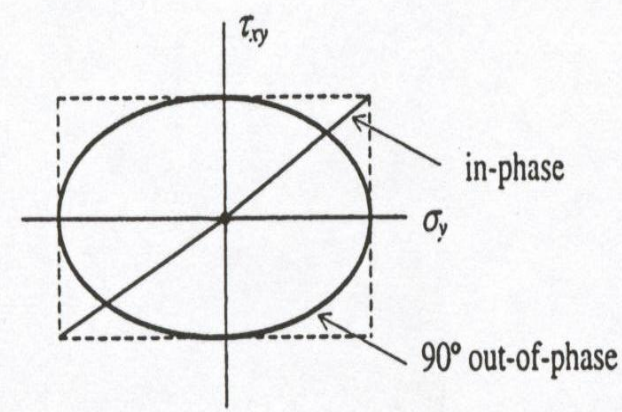
- If the loads are applied 90 **out-of-phase** (Fig. *d*), the stress path, σ_y-τ_{xy}, follows an ellipse, as shown in Fig. *c*. The ratio of axial stress, σ_y, and torsion stress, τ_{xy}, continuously varies during the cycle.
  - This is, therefore, an example of **nonproportional** loading.
  - Mohr's circles of stress at three times (1, 2, and 3) during the out-of-phase loading cycle are shown in Fig. *f*. The orientations of the principal normal stress axes continuously rotate with respect to loading axes (i.e. *x*-*y* axes).



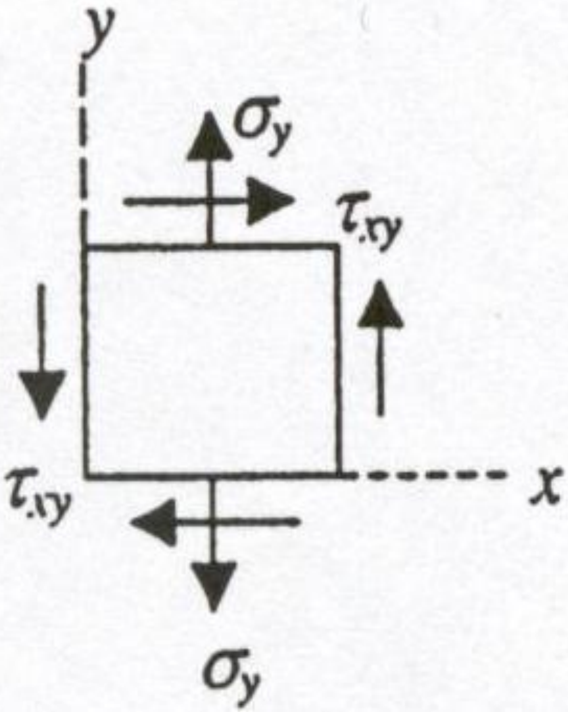
(*a*)

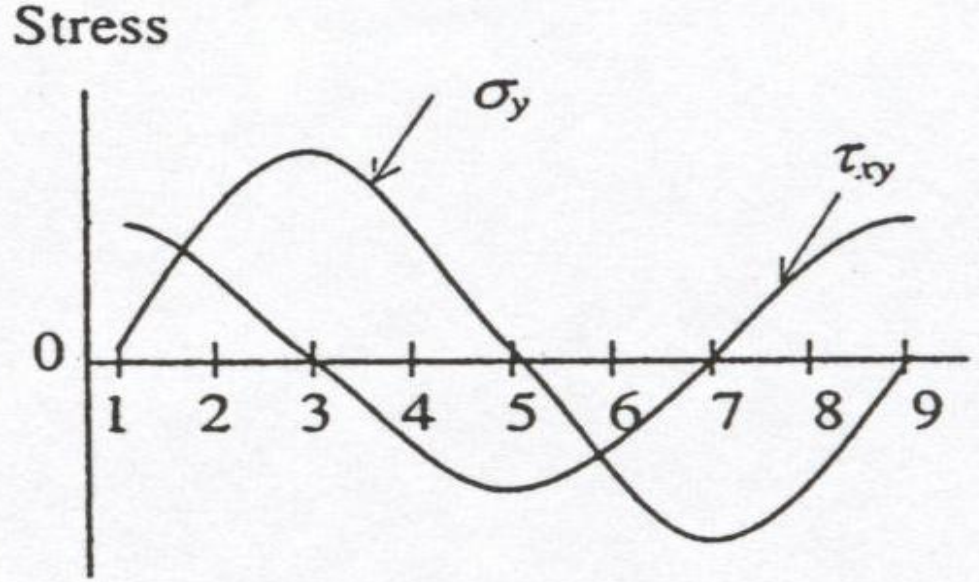


(*d*)

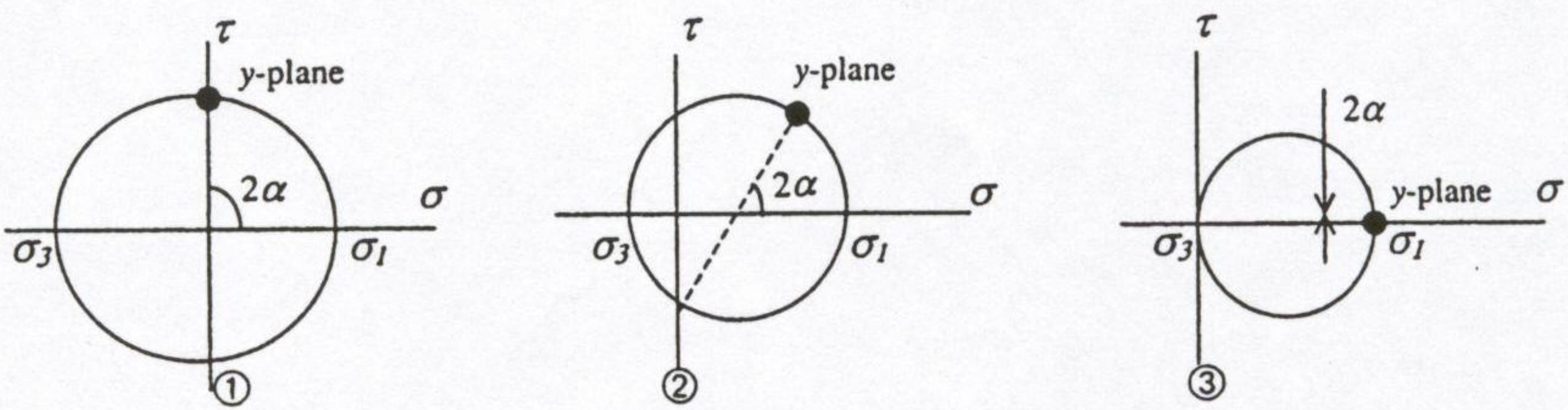


(*c*)


y
σ_y
τ_{xy}
τ_{xy}
x
σ_y


Stress
σ_y
τ_{xy}
0
t
1
2
3
4
5
6
7
8
9

(*d*)


τ
y-plane
2α
σ
σ_3
σ_1
①
τ
y-plane
2α
σ
σ_3
σ_1
②
τ
2α
y-plane
σ
σ_3
σ_1
③

(*f*)

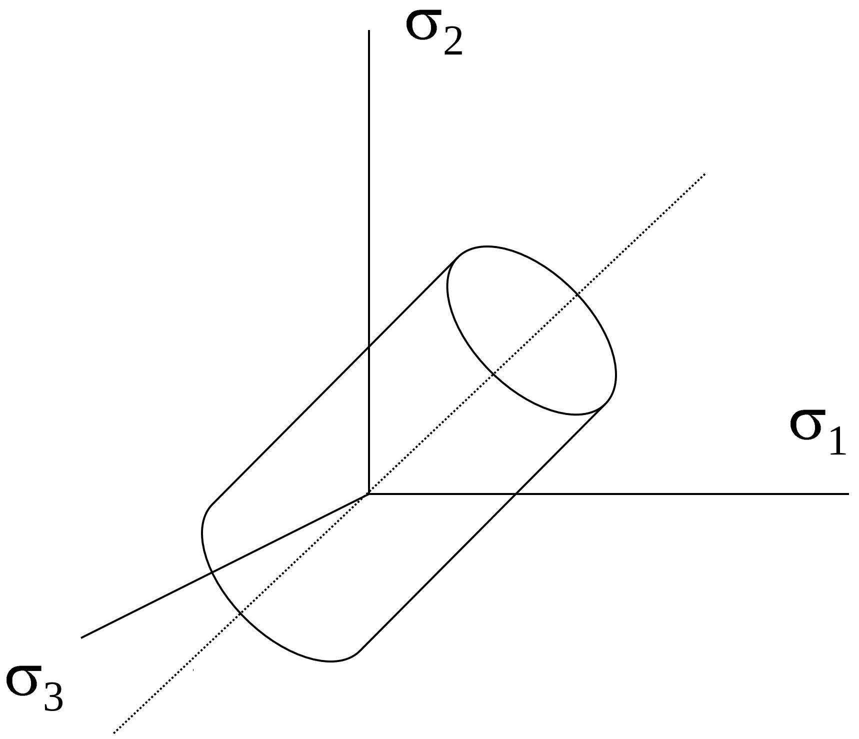
MULTIAXIAL STRESSES
(YIELDING AND PLASTICITY)

- Cyclic plastic deformation is an essential component of the fatigue damage process. Therefore, an understanding of **multiaxial cyclic plastic deformation** is often necessary, particularly in situations where significant plasticity exists such as at notches and in low cycle fatigue.

- The basic elements of plasticity theory consist of
  - a **yield function** to determine when plastic flow initiates,
  - a **flow rule** which relates the applied stress increments to the resulting plastic strain increments once plastic flow has initiated,
  - a **hardening rule** that describes the change in the yield criterion as a function of plastic strains.

MULTIAXIAL STRESSES
(YIELDING AND PLASTICITY)

- A commonly used yield criterion for metals is the **von Mises** yield criterion.
  - It can be visualized as a circular cylinder in the stress space.
  - For unyielded material the axis of the cylinder passes through the origin of the coordinates. It is inclined equal amounts to the three coordinate axes and represents pure hydrostatic stress (i.e. $\sigma_1 = \sigma_2 = \sigma_3$).



MULTIAXIAL STRESSES
(YIELDING AND PLASTICITY)

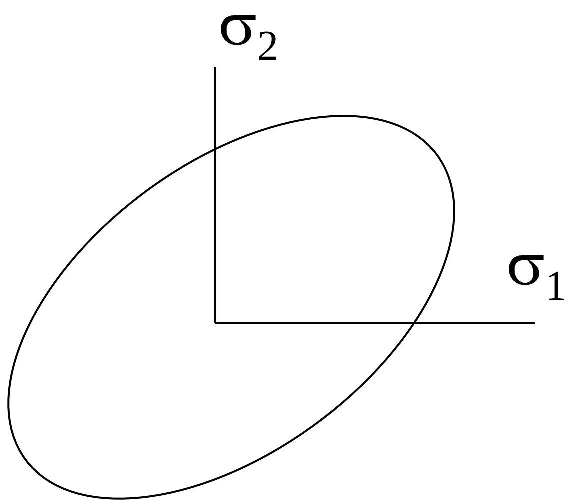
- The **von Mises yield criterion** is given by

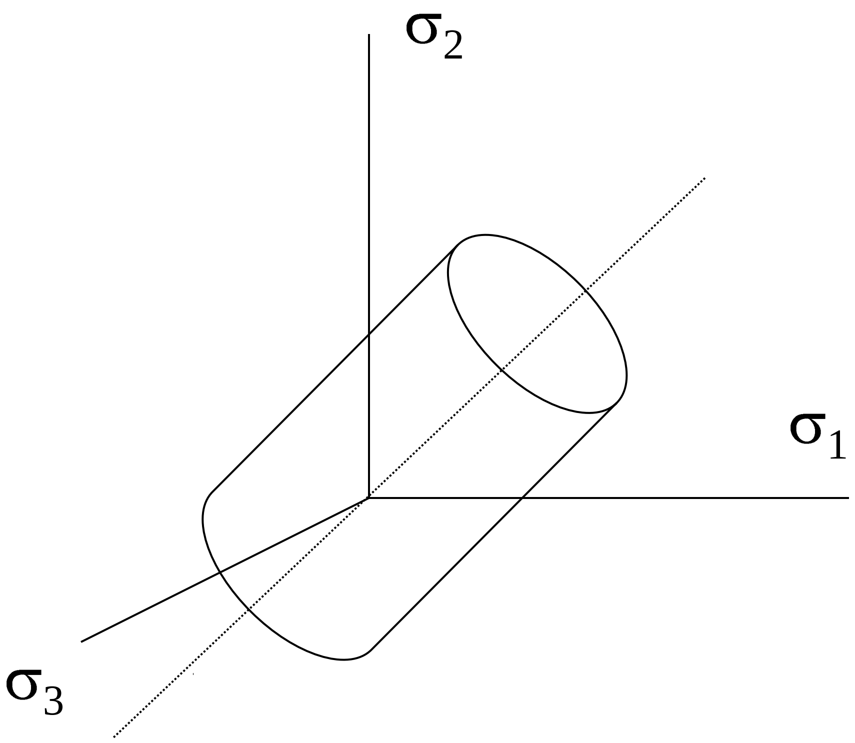
$$\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = 2S_y^2$$

where S_y is the material yield strength.

MULTIAXIAL STRESSES
(YIELDING AND PLASTICITY)

- For **biaxial** or plane states of stress ($\sigma_3 = 0$), the yield condition is the intersection of the cylinder with the σ_1-σ_2 plane, which is a **yield ellipse**.





MULTIAXIAL STRESSES
(YIELDING AND PLASTICITY)

- It is often convenient to convert the multiaxial stress state to an "**equivalent**" stress σ_e, which is the uniaxial stress that is equally distant from (or located on) the yield surface.
  - During initial loading, the von Mises equivalent stress is given by:

$$\sigma_e = \frac{1}{\sqrt{2}}\sqrt{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}$$

or $$\sigma_e = \frac{1}{\sqrt{2}}\sqrt{(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2+6(\tau^2_{xy}+\tau^2_{yz}+\tau^2_{zx})}$$

where σ_x, τ_{xy}, and so on are stresses in an arbitrary orthogonal coordinate system.

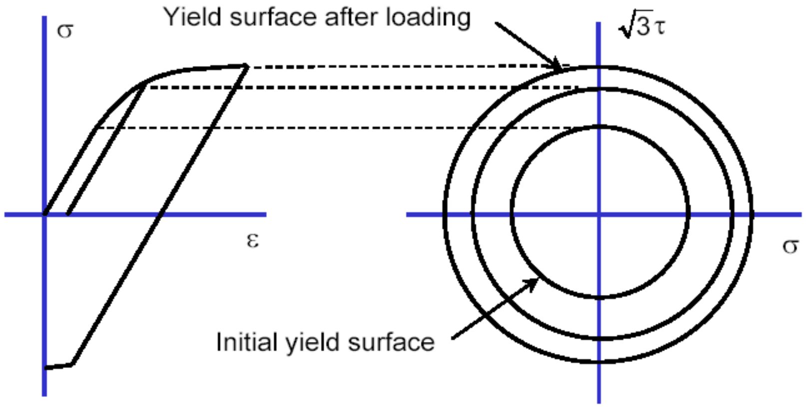
  - Yielding occurs when $\sigma_e = S_y$.
  - These expressions for the equivalent stress σ_e, are also related to the octahedral shear stress and the distortion energy.

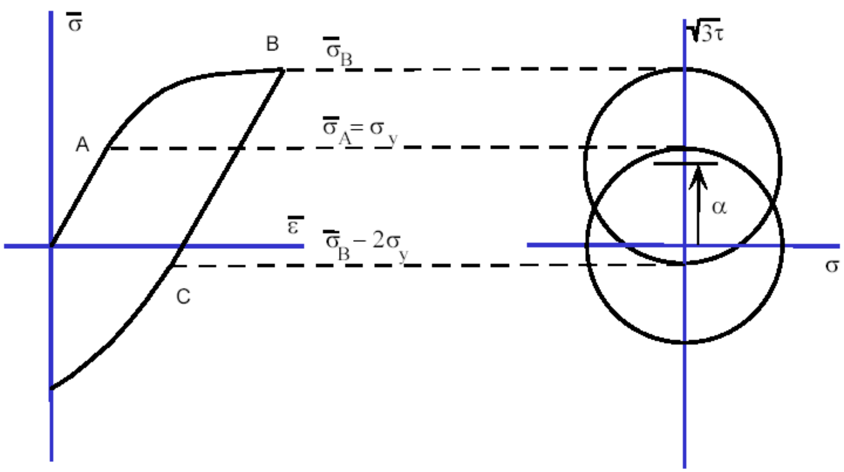
MULTIAXIAL STRESSES
(YIELDING AND PLASTICITY)

- Once plastic deformation has initiated, we need a **flow rule** to relate stresses and plastic strains.
  - Equations relating stresses and plastic strains are also called **constitutive equations**.
  - They are typically based on the **normality condition**. This condition states that the increment of plastic strain caused by an increment of stress is such that the vector representing the plastic strain increment is normal to the yield surface during plastic deformation.

MULTIAXIAL STRESSES
(YIELDING AND PLASTICITY)

- A **hardening rule** is needed to describe the behavior of the material once it is plastically deformed or yielded.
  - One possible hardening rule is the **isotropic rule**, which assumes that strain hardening corresponds to an enlargement of the yield surface (i.e. an increase in S_y) without change of shape or position in the stress space.
  - Another is the **kinematic rule**, which assumes that strain hardening shifts the yield surface without changing its size or shape.

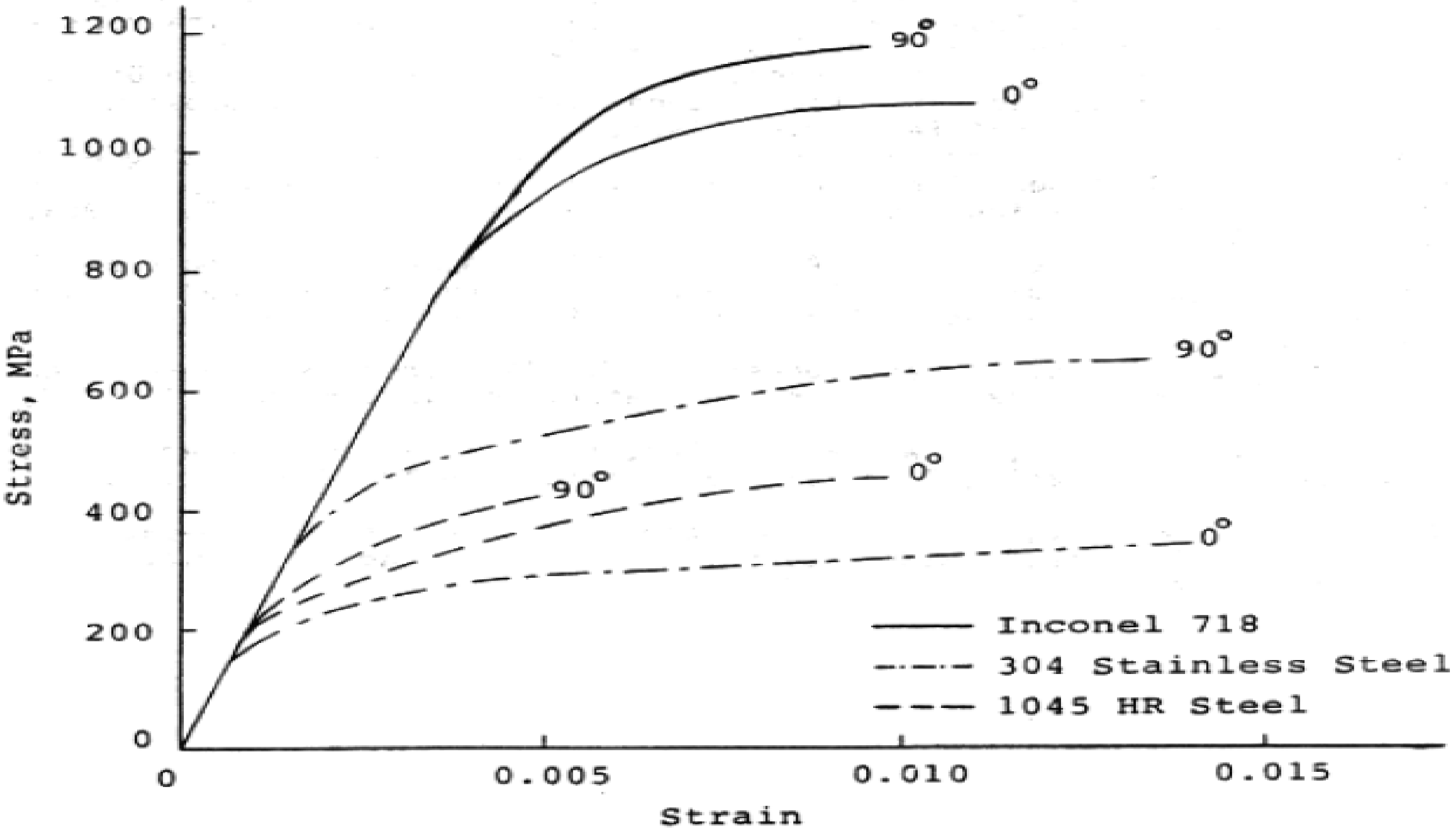




MULTIAXIAL STRESSES
(YIELDING AND PLASTICITY)

- Nonproportional multiaxial cyclic loading often produces **additional strain hardening** which is not observed under proportional loading conditions.
  - Therefore, the cyclic stress-strain curve for nonproportional loading is above that for proportional loading.
  - The reason for the additional hardening is the **interaction of slip planes**, since many more slip planes are active during nonproportional loading due to the rotation of the principal axes.
  - The amount of this additional hardening depends on the **degree of load nonproportionality** as well as the material.

ADDITIONAL STRAIN HARDENING IN NONPROPORTIONAL LOADING



MULTIAXIAL FATIGUE
LIFE ESTIMATION METHODS

- Stress-Based Approaches,
- Strain-based & Energy-Based Approaches,
- Critical Plane Models, and
- Fracture Mechanics Approach for Crack Growth.

MULTIAXIAL STRESSES
(STRESS-BASED CRITERIA)

- Equivalent Stress Approaches
- Sines Method
- Examples Using Stress-Life Approach

MULTIAXIAL STRESSES
(STRESS-BASED CRITERIA)

- **Equivalent Stress Approaches**
  - Equivalent stress approaches are extensions of **static yield criteria** to fatigue.
  - The most commonly used equivalent stress approaches for fatigue are the maximum principal stress theory, the maximum shear stress theory (or Tresca), and the octahedral shear stress theory (or von Mises).
  - An "**equivalent**" nominal stress amplitude, S_{qa}, can be computed according to each criterion.

MULTIAXIAL STRESSES
(STRESS-BASED CRITERIA)

- **Maximum principal stress** theory: $S_{qa} = S_{a1}$
- **Maximum shear stress** theory: $S_{qa} = S_{a1} - S_{a3}$
- **Octahedral shear stress** theory (identical to **von Mises** equivalent stress for static loading, except that alternating stresses are used):

$$S_{qa} = \frac{1}{\sqrt{2}}\sqrt{(S_{a1} - S_{a2})^2 + (S_{a2} - S_{a3})^2 + (S_{a3} - S_{a1})^2}$$

- S_{a1}, S_{a2}, and S_{a3} are principal alternating nominal stresses with $S_{a1} > S_{a2} > S_{a3}$.

MULTIAXIAL STRESSES
(STRESS-BASED CRITERIA)

- Once the equivalent nominal stress amplitude, S_{qa}, is calculated, the multiaxial stress state is reduced to an equivalent uniaxial stress state.

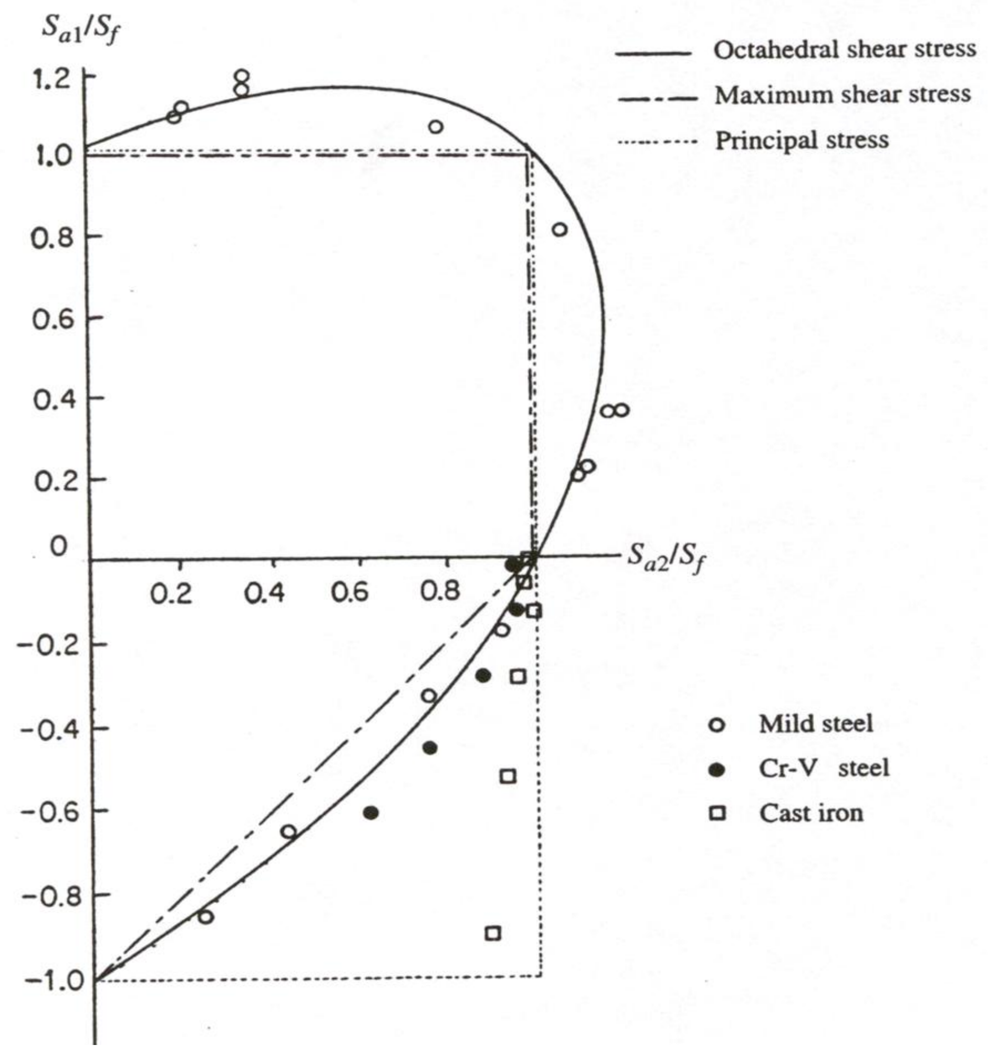
- Therefore, the *S-N* approach discussed in Chapter 4 can then be used for fatigue life calculations by setting S_{qa} equal to the appropriate fatigue strength S_{Nf} or S_f.

- The **octahedral shear stress criterion (von Mises)** is the most widely used equivalent stress criterion for multiaxial fatigue of materials having a **ductile** behavior.

- The **maximum principal stress** criterion is usually better for multiaxial fatigue of materials with **brittle** behavior.

MULTIAXIAL STRESSES (STRESS-BASED CRITERIA)

- The three aforementioned criteria are shown for biaxial stress states ($S_3 = 0$).
  - data for the mild steel and Cr-V steel, which behave in a ductile manner, agree well with the octahedral shear stress criterion.
  - data for cast iron, which behaves in a brittle manner, agrees better with the maximum principal stress criterion.



MULTIAXIAL STRESSES
(STRESS-BASED CRITERIA)

- If **mean or residual stresses** are present,
  - An equivalent mean nominal stress, S_{qm}, can be calculated based on the **von Mises** effective stress

$$S_{qm} = \frac{1}{\sqrt{2}}\sqrt{(S_{m1} - S_{m2})^2 + (S_{m2} - S_{m3})^2 + (S_{m3} - S_{m1})^2} \qquad (10.10)$$

  - Another commonly used equivalent mean stress is the **sum of mean normal stresses**

$$S_{qm} = S_{m1} + S_{m2} + S_{m3} = S_{mx} + S_{my} + S_{mz} \qquad (10.11)$$

Note that the second equality exists because the sum of normal stresses represents a stress invariant (i.e. independent of coordinate axes used).

S_{m1}, S_{m2}, and S_{m3} are principal mean nominal stresses.

MULTIAXIAL STRESSES
(STRESS-BASED CRITERIA)

- Tresca and von Mises equivalent stresses are insensitive to **hydrostatic stress**.
  - Therefore, if the mean stress is a hydrostatic stress, Eq. 10.10 results in $S_{qm} = 0$.
  - Since fatigue life has been observed to be sensitive to hydrostatic stress, the use of Eq. 10.11 is preferred for this case.

- Eq. 10.10 would always result in a positive equivalent mean stress, whereas Eq. 10.11 can be either positive or negative. Equation 10.11, therefore, better represents the beneficial effect of compressive mean stress and the detrimental effect of tensile mean stress on fatigue life.

MULTIAXIAL STRESSES
(STRESS-BASED CRITERIA)

- According to **Eq. 10.11**,

$$S_{qm} = S_{m1} + S_{m2} + S_{m3} = S_{mx} + S_{my} + S_{mz}$$

  - mean torsion stress has no direct influence on fatigue (i.e. $S_{qm} = 0$ for mean torsion). This agrees with experimental evidence, as long as the maximum shear stress remains below yielding for the material.

  - the effect of a tensile mean stress acting in one direction can be nullified by a compressive mean stress acting in another direction which does not necessarily agree with experimental evidence.

MULTIAXIAL STRESSES (STRESS-BASED CRITERIA)

- Stresses, S_{qa} and S_{qm}, are those equivalent alternating and mean stresses that can be expected to give the same life in uniaxial loading as in multiaxial loading.

- After S_{qa} and S_{qm} are calculated the expected fatigue life is found from the formulas for uniaxial fatigue, such as the modified Goodman equation.

- Equivalent stress approaches
  - have been commonly used because of their **simplicity**, but their success in correlating multiaxial fatigue data has been limited to a few materials and loading conditions.
  - should only be used for **proportional loading** conditions, where the principal axes directions remain fixed during the loading cycle.

MULTIAXIAL STRESSES
(STRESS-BASED CRITERIA)

- ***Sines Method***
  - Sines method uses alternating octahedral shear stress for cyclic stresses and hydrostatic stress for mean stresses

$$\sqrt{(S_{a1} - S_{a2})^2 + (S_{a2} - S_{a3})^2 + (S_{a3} - S_{a1})^2} + m\,(S_{mx} + S_{my} + S_{mz}) = \sqrt{2}\,S_{Nf}$$

- S_{Nf} is the uniaxial fully reversed fatigue strength that is expected to give the same fatigue life on uniaxial smooth specimens as the multiaxial stress state.
- ***m*** is the coefficient of mean stress influence.
  - It can be determined experimentally by obtaining a fatigue strength with a nonzero mean stress level (i.e. for example, uniaxial fatigue strength for R = 0 condition where $S_m = S_a$).
  - The value of m is on the order of 0.5

MULTIAXIAL STRESSES
(STRESS-BASED CRITERIA)

- In terms of *x, y, z* stress components, **Sines method** is expressed by the equation

$$\sqrt{(S_{ax}-S_{ay})^2+(S_{ay}-S_{az})^2+(S_{az}-S_{ax})^2+6(\tau^2_{axy}+\tau^2_{ayz}+\tau^2_{azx})}+m(S_{mx}+S_{my}+S_{mz})=\sqrt{2}\,S_{Nf}$$

- Sines method should also be limited to **proportional loading**. For this type of loading, this theory fits most observations for **long life** fatigue and can be extended for application to strain-controlled low cycle fatigue.

- It should be recognized that stress-based approaches are, in general, suitable for **long life** fatigue situations, where strains are mainly elastic.

Examples Using
Stress-Life Approach

involving proportional biaxial loading

MULTIAXIAL STRESSES
(STRAIN-BASED APPROACHES)

- Strain-based approaches are used with the strain-life curve in situations where significant **plastic deformation** can exist such as in low cycle fatigue or at notches.

- Analogous to equivalent stress approaches, **equivalent strain approaches** have been used as strain-based multiaxial fatigue criteria.

- Similar to the equivalent stress approaches, equivalent strain approaches are also not suitable for **nonproportional multiaxial loading** situations.

MULTIAXIAL STRESSES
(STRAIN-BASED APPROACHES)

- The most **commonly used equivalent strain approaches** are strain versions of the equivalent stress models as follows

  - Maximum principal strain theory: $\varepsilon_{qa} = \varepsilon_{a1}$

  - Maximum shear strain theory: $$\varepsilon_{qa} = \frac{\varepsilon_{a1} - \varepsilon_{a3}}{1+\nu}$$

  - Octahedral shear strain theory:

$$\varepsilon_{qa} = \frac{\sqrt{(\varepsilon_{a1} - \varepsilon_{a2})^2 + (\varepsilon_{a2} - \varepsilon_{a3})^2 + (\varepsilon_{a3} - \varepsilon_{a1})^2}}{\sqrt{2}\,(1+\nu)}$$

ε_{a1}, ε_{a2}, and ε_{a3} are principal alternating strains with $\varepsilon_{a1} > \varepsilon_{a2} > \varepsilon_{a3}$.

- Once an equivalent alternating strain, ε_{qa}, has been calculated from the multiaxial stress state, the strain-life equation is used for life prediction.

MULTIAXIAL STRESSES
(ENERGY-BASED APPROACHES)

- **Energy-based** approaches use products of stress and strain to quantify fatigue damage.

- Several energy quantities have been proposed for multiaxial fatigue, such as:
  - **Plastic work per cycle** as the parameter for life to crack nucleation.
    - Plastic work is calculated by integrating the product of stress times plastic strain increment (the area of the hysteresis loop) for each of the six components of stress.
    - The sum of the six integrals is the plastic work per cycle.
    - Application of this method to high cycle fatigue situations is difficult since plastic strains are small.
  - **Total strain energy density per cycle**, consisting of both elastic and plastic energy density terms.

MULTIAXIAL STRESSES
(ENERGY-BASED APPROACHES)

- Energy-based approaches can be used for nonproportional loading.

- Energy, however, is a **scalar quantity** and, therefore, does not reflect fatigue damage nucleation and growth observed on specific planes.

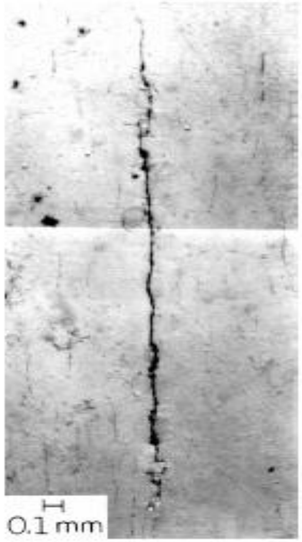
DAMAGE MECHANISMS

- Cracks **nucleate** in shear, but **grow** in shear or tension depending on the material and stress state.

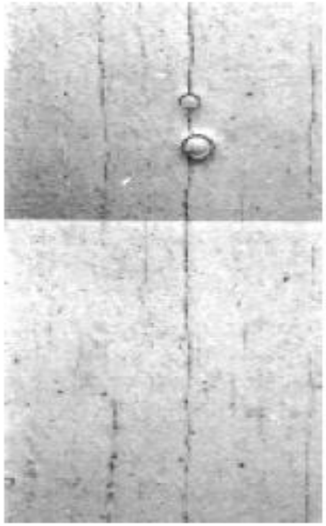
Initiation/nucleation → Early/small crack growth

(microscopic, on the order of grain size) (macroscopic, ≈ a few mm)

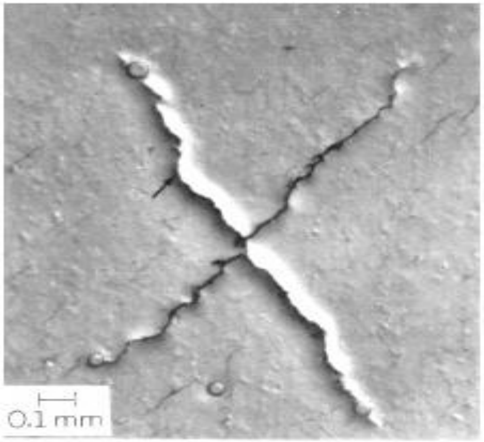
Damage Mechanisms Comparisons in **Torsion**



Inconel



1045 steel



stainless steel

MULTIAXIAL STRESSES
(CRITICAL PLANE APPROACHES)

- Experimental observations indicate that cracks nucleate and grow on specific planes (also called **critical planes**).
- Depending on the material and loading conditions, these planes are either maximum shear planes or maximum tensile stress planes.
- Multiaxial fatigue models relating fatigue damage to stresses and/or strains on these planes are called **critical plane models**.
- These models, therefore, not only can predict the fatigue life, but also the **orientation of the crack or failure plane**.
- Different damage parameters using stress, strain, or energy quantities have been used to evaluate damage on the critical plane.

MULTIAXIAL STRESSES
(FATEMI-SOCIE CRITICAL PLANE APPROACH)

- Critical plane approaches attempt to reflect the **physical nature of fatigue damage** (i.e. damage mechanisms) in their formulation.

- A common model based on the physical interpretation of fatigue damage is the **Fatemi-Socie model**.

  - In this model, the parameters governing fatigue damage are the maximum shear strain amplitude, $\Delta\gamma_{max}/2$, and the maximum normal stress, $\sigma_{n,max}$, acting on the maximum shear strain amplitude plane.

  - The physical basis for this model is illustrated in Fig. 10.3.

MULTIAXIAL STRESSES
(FATEMI-SOCIE CRITICAL PLANE APPROACH)

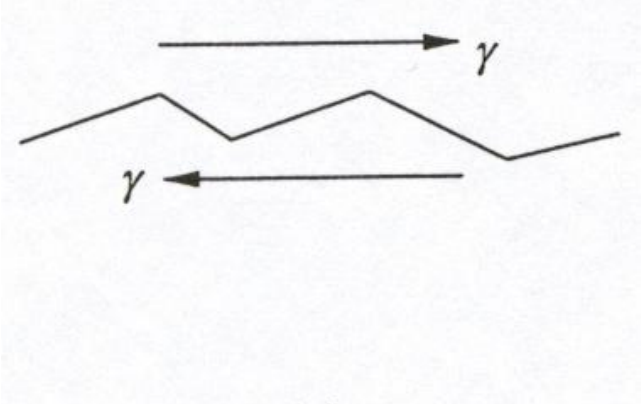
- Cracks are usually irregularly shaped at the microscopic level, as the crack grows through the material grain structure.

- This results in interlocking and friction forces between crack surfaces during the cyclic shear loading (i.e. crack closure), as shown in Fig. *a*.

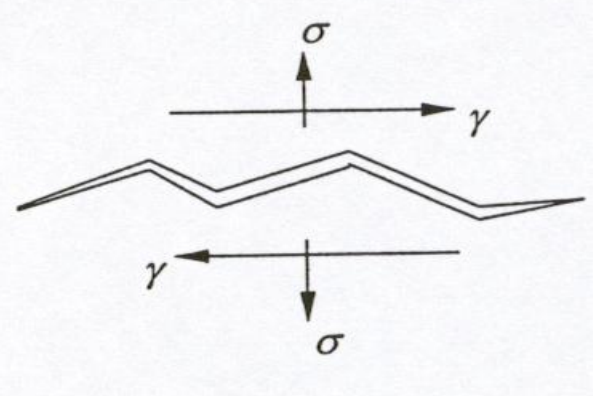
- Consequently, the crack tip driving force is reduced and the fatigue life is increased.

- A tensile stress perpendicular to the crack plane tends to separate crack surfaces, and therefore, reduce interlocking and frictional forces, as shown in Fig. *b*. This increases the crack tip driving force and the fatigue life is reduced.

- Fractographic evidence of this behavior has been observed.



(*a*)



(*b*)

MULTIAXIAL STRESSES
(FATEMI-SOCIE CRITICAL PLANE APPROACH)

- The Fatemi-Socie model is expressed as

$$\frac{\Delta\gamma_{\max}}{2}\left(1+k\frac{\sigma_{n,\max}}{S_y}\right)=C$$

  - The maximum normal stress is normalized by the material monotonic yield strength to preserve the unitless feature of strain.
  - *k* is a material constant.
    - Its value can be found by fitting fatigue data from simple uniaxial tests to fatigue data from simple torsion tests (note that for simple torsion test, $\sigma_{n,max}$ = 0, and the left side of the equation reduces to $\Delta\gamma_{max}/2$).
    - As a first approximation, or if test data are not available, $k \approx 1$.

MULTIAXIAL STRESSES
(FATEMI-SOCIE CRITICAL PLANE APPROACH)

- **Mean or residual stress effects** on fatigue life in this model are accounted for by the maximum normal stress term, since:

$$\sigma_{n,max} = \sigma_{n,a} + \boldsymbol{\sigma_{n,m}}$$

where $\sigma_{n,a}$ and $\sigma_{n,m}$ are the alternating normal and mean or residual normal stresses, respectively.

- **Additional hardening** resulting from nonproportional loading is also incorporated by the maximum normal stress term since additional hardening results in an increase in the alternating normal stress, $\sigma_{n,a}$.

MULTIAXIAL STRESSES
(FATEMI-SOCIE CRITICAL PLANE APPROACH)

- The Fatemi-Socie equation can be written in terms of shear strain-life properties obtained from fully reversed torsion tests (usually using thin-walled tube specimens) as

$$\frac{\Delta\gamma_{\max}}{2}\left(1+k\frac{\sigma_{n,\max}}{S_y}\right)=\frac{\tau'_f}{G}(2N_f)^{b_0}+\gamma'_f(2N_f)^{c_0}$$

  - G is the shear modulus,
  - τ'_f is the shear fatigue strength coefficient,
  - γ'_f is the shear fatigue ductility coefficient, and
  - b_o and c_o are shear fatigue strength and shear fatigue ductility exponents, respectively.
  - If these properties are not available, they can be estimated from uniaxial strain-life properties as: $\tau'_f \approx \sigma'_f/\sqrt{3}$, $b_o \approx b$, $\gamma'_f \approx \sqrt{3}\varepsilon'_f$, and $c_o \approx c$.

MULTIAXIAL STRESSES
(FATEMI-SOCIE CRITICAL PLANE APPROACH)

- This equation can also be expressed in terms of the uniaxial strain-life properties by equating the left side of the equation to *C* for fully reversed uniaxial straining

$$\frac{\Delta\gamma_{max}}{2}\left(1+k\frac{\sigma_{n,max}}{S_y}\right)=\left[(1+\nu_e)\frac{\sigma'_f}{E}(2N_f)^b+(1+\nu_p)\varepsilon'_f(2N_f)^c\right]\left[1+k\frac{\sigma'_f}{2S_y}(2N_f)^b\right]$$

where ν_e and ν_p are elastic and plastic Poisson's ratios, respectively.

- Once the left side of either equation has been calculated for the multiaxial loading condition, fatigue life can be obtained from the right side.

MULTIAXIAL STRESSES
(FATEMI-SOCIE CRITICAL PLANE APPROACH)

- Figure 10.4 illustrates correlation of multiaxial fatigue data for Inconel 718 using thin-walled tube specimens, based on the Fatemi-Socie parameter.

- The data shown represent a **wide variety of constant amplitude loading conditions** obtained from:

  - in-phase (proportional) axial-torsion tests with or without mean stress,
  - out-of-phase (nonproportional) axial-torsion tests, and
  - biaxial tension tests (using internal/external pressurization) with or without mean stress.

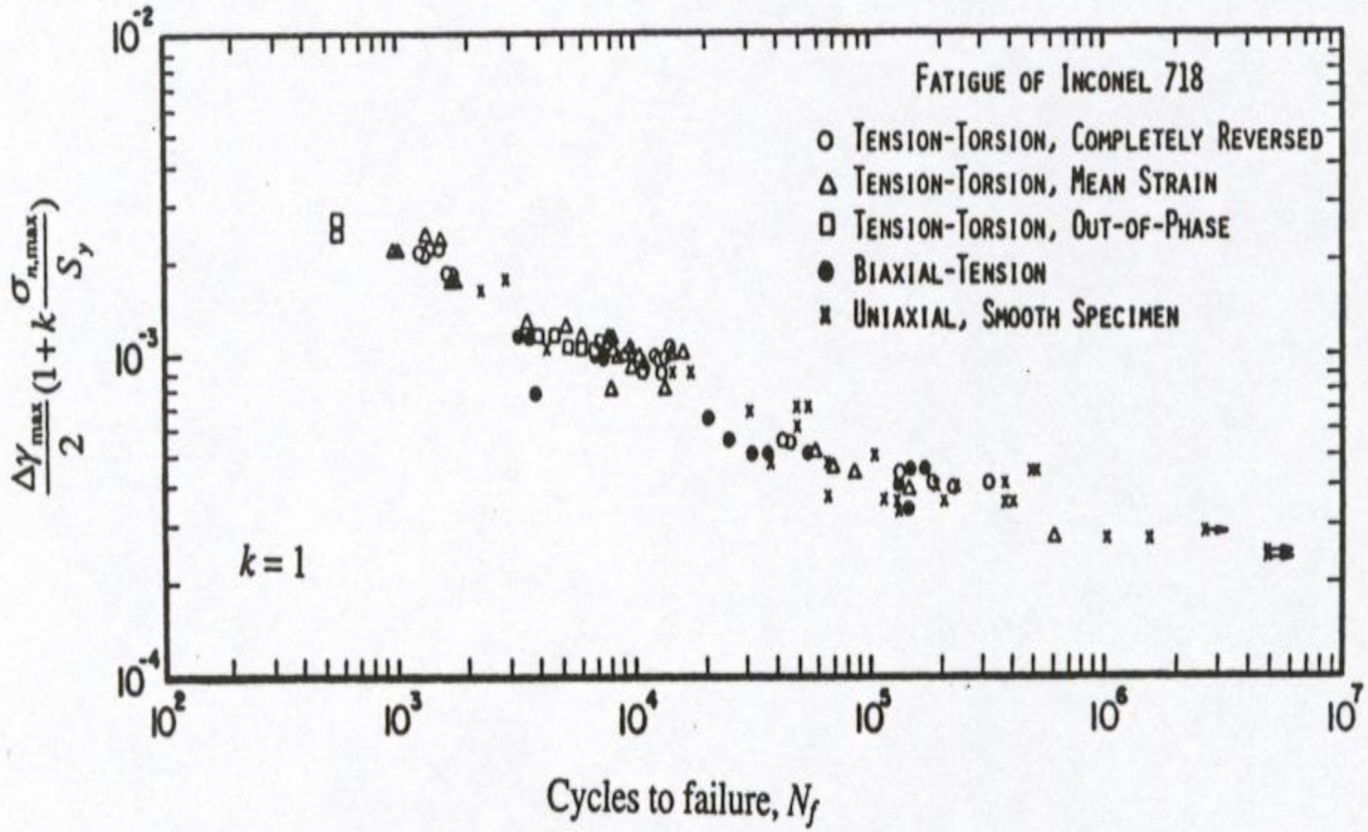


Figure 10.4 Correlation of Inconel 718 multiaxial fatigue data using the Fatemi-Socie parameter [10] (reprinted by permission of D. L. Morrow).

MULTIAXIAL STRESSES
(FATEMI-SOCIE CRITICAL PLANE APPROACH)

- Park and Nelson have evaluated this model under a wide variety of proportional and nonproportional constant amplitude loading conditions for a **variety of metallic alloys** including:

  - mild steels,
  - high strength and high temperature steel alloys,
  - stainless steels, and
  - several aluminum alloys

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Evaluation of an energy-based approach and a critical plane approach for predicting constant amplitude multiaxial fatigue life

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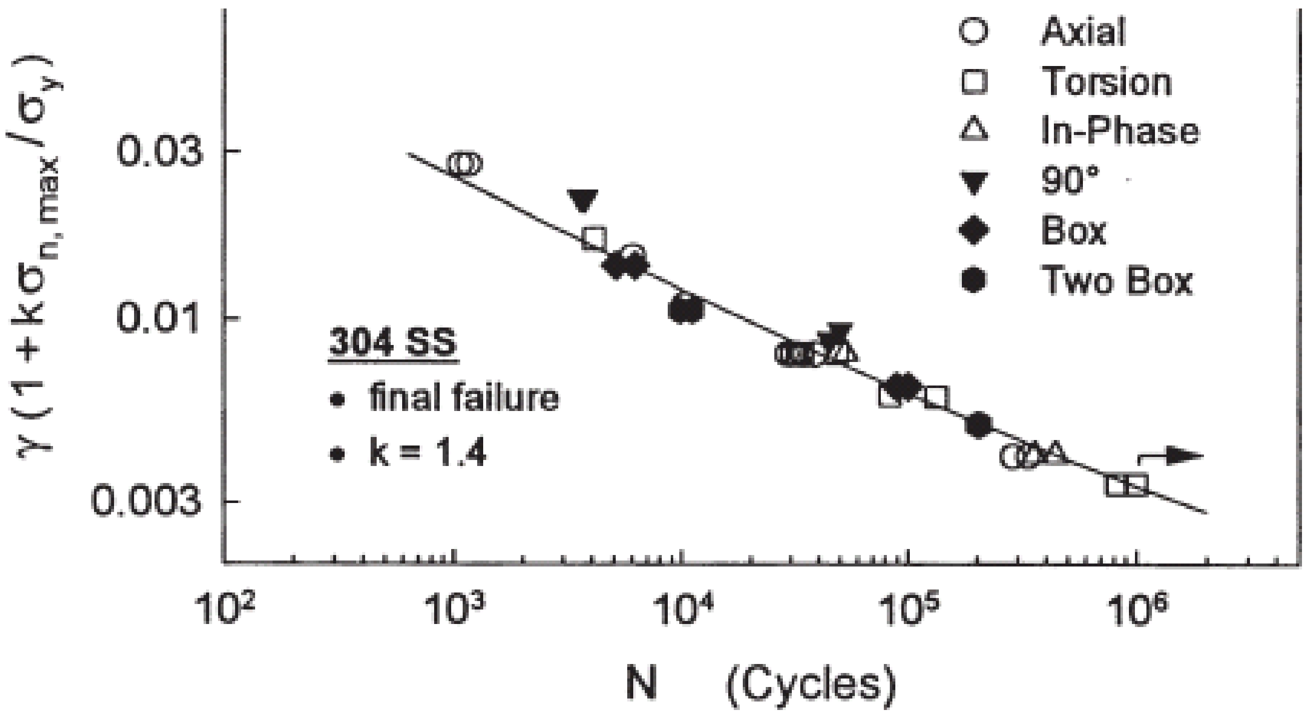
Material	n^a
SAE 1045 steel	49
A533B steel	57
A533B steel	66
Mild steel	29
Inconel 718	60
Inconel 718	69
30Cr-Mi-Mo 8 steel	37
A-516 Gr.70 steel	37
304 stainless steel	33
Haynes 188	16
X 10 Cr-Ni-Ti 18 9	29
304 stainless steel	33
1% Cr-Mo-V steel	54
1% Cr-Mo-V steel	16
Al-7475	14
Al-6061-T6 (Rod 1)	17
Al-6061-T6 (Rod 2)	14
Al-6061-T6 (RD)	16
Al-6061-T6 (TD)	16
Al-7075-T6	56

[a] n is the number of data paints.



MULTIAXIAL STRESSES
(FATEMI-SOCIE CRITICAL PLANE APPROACH)

- According to the Fatemi-Socie model, cyclic shear strain must be present for fatigue damage to occur (the equation indicates no damage, if $\Delta\gamma_{max}/2 = 0$).
- Therefore, this model is suitable for materials where the majority of the fatigue life is spent in crack nucleation and small crack growth along the **maximum shear** planes.
- This is representative of many metals and alloys. For some materials such as cast iron, however, crack nucleation and/or crack growth is along the maximum tensile stress or strain planes.

MULTIAXIAL STRESSES
(FATEMI-SOCIE CRITICAL PLANE APPROACH)

- In this case, the Smith-Watson-Topper parameter can be used as the damage model, where

  - The governing parameters are the maximum principal strain amplitude, ε_{a1}, and the maximum normal stress on the maximum principal strain amplitude plane, $\sigma_{n,max}$.

$$\sigma_{\max}\,\varepsilon_a E = \left(\sigma'_f\right)^2 \left(2N_f\right)^{2b} + \sigma'_f\,\varepsilon'_f\,E \left(2N_f\right)^{b+c}$$

  - The critical plane is the plane with the largest value of ($\varepsilon_{a1}\sigma_{n,max}$).

  - Similar to the Fatemi-Socie model, mean and residual stress effects and additional hardening due to nonproportional loading are incorporated through the maximum normal stress term.

MULTIAXIAL STRESSES
(**FRACTURE MECHANICS** FOR FATIGUE CRACK GROWTH)

- Fracture mechanics is used to characterize the growth of cracks, where multiaxial stress states can result in **mixed mode crack growth**.
- A characteristic of mixed mode fatigue cracks is that they can grow in a **non-self similar** manner, that is, the crack changes its growth direction.
- Therefore, under mixed mode loading conditions, both **crack growth direction and crack growth rate** are important.

MULTIAXIAL STRESSES
(**FRACTURE MECHANICS** FOR FATIGUE CRACK GROWTH)

- Different combinations of mixed mode loading can exist from multiaxial loads. For **example**,
  - In simple **torsion of smooth shafts**,
    - Surface cracks can form and grow in longitudinal and/or transverse directions along the maximum shear planes, where mixed mode II and III exits along the crack front.
    - Cracks can also form and grow along the $\pm 45°$ to the axis of the shaft along planes of maximum principal stress where they grow in mode I.
  - Plate components with edge or central cracks under in-plane biaxial tension or under three or four point bending and shear loading often produce mixed mode I and II crack growth. In plate components, however, mode I contribution often becomes dominant after a short period of crack growth.

MULTIAXIAL STRESSES
(**FRACTURE MECHANICS** FOR FATIGUE CRACK GROWTH)

- Several parameters have been used to correlate fatigue crack growth rates under mixed mode conditions. For example,
  - an equivalent stress intensity factor range, ΔK_q, based on crack tip displacements proposed by Tanaka is given by

$$\Delta K_q = \left(\Delta K_I^{\,4} + 8\,\Delta K_{II}^{\,4} + \frac{8\,\Delta K_{III}^{\,4}}{1-\nu} \right)^{0.25}$$

  - Another form of equivalent stress intensity factor is given by

$$\Delta K_q = \left[\Delta K_I^{\,2} + \Delta K_{II}^{\,2} + (1+\nu)\,\Delta K_{III}^{\,2} \right]^{0.5}$$

- ΔK_q has been used for the mixed mode loading condition in a Paris type equation to obtain the crack growth rate, *da/dN*, or cycles to failure through integration.

MULTIAXIAL STRESSES
(NOTCH EFFECTS)

- The stress state at the root of the notch is often multiaxial, even under uniaxial loading.
  - For example, in axial loading of a circumferentially notched bar, both axial and tangential stresses exist at the root of the notch.
  - The tangential stress component results from the notch constraint in the transverse direction.

- In the *S-N* approach, fatigue strength can be divided by the fatigue notch factor, K_f.
  - The stress concentration factor not only depends on the notch geometry, but also on the type of loading.
  - If the theoretical stress concentration factors, K_t's, for multiaxial loading differ too greatly for different principal directions, each nominal alternating stress component can be multiplied by its corresponding fatigue notch factor, K_f.

MULTIAXIAL STRESSES
(NOTCH EFFECTS)

- This can be done because the *S-N* approach is used for mainly elastic behavior and, therefore, superposition can be used to estimate notch stresses from combined multiaxial loads.

- For example, for combined bending and torsion of a notched shaft and in the absence of mean stresses, an equivalent stress based on the octahedral shear stress can be computed as

$$S_{qa} = \sqrt{\left(K_{fB}\, S_B\right)^2 + 3\left(K_{fT}\, S_T\right)^2}$$

where S_B and S_T are nominal bending and torsion stresses, respectively, and K_{fB} and K_{fT} are fatigue notch factors in bending and torsion, respectively.

MULTIAXIAL STRESSES
(NOTCH EFFECTS)

- Notch effects in the ε-N approach for multiaxial loading when notch root plastic deformation exists is more complex and often requires the use of cyclic plasticity models.

- For this case, Neuber's rule is often generalized to the multiaxial loading situation by using equivalent stresses and strains.

MULTIALIAL STRESSES
(VARIABLE AMPLITUDE LOADING)

- Multiaxial fatigue analysis for variable amplitude loading is quite complex, particularly when the applied loads are out-of-phase or nonproportional.
- Similar to the uniaxial loading, the two main steps associated with cumulative damage analysis from multiaxial variable amplitude loading are identification or definition of a cycle, and evaluation of damage for each identified or counted cycle.
- Several methods for predicting cumulative damage in multiaxial fatigue have been developed, such as those by Bannantine and Socie, and Wang and Brown.

MULTIAXIAL STRESSES (SUMMARY)

- Multiaxial states of stress are very **common** in engineering components and structures.

- Multiaxial cyclic loading can be **categorized** as proportional where the orientation of the principal axes remains fixed, or nonproportional where the principal directions of the alternating stresses change orientation during cycling.

- Nonproportional cyclic loading involves greater difficulties and often produces **additional cyclic hardening**, as compared to the proportional loading.

MULTIAXIAL STRESSES (SUMMARY)

- Multiaxial fatigue analysis for situations where significant plasticity exists often requires the use of a cyclic plasticity model consisting of a yield function, a flow rule, and a hardening rule.

- Stress-based, strain-based, energy-based, and critical plane approaches have been used for life prediction under multiaxial stress states.

- The most common stress-based methods are those based on equivalent octahedral shear stress and Sines method.

MULTIAXIAL STRESSES (SUMMARY)

- Equivalent nominal stress amplitude based on the octahedral shear stress criterion and equivalent mean nominal stress based on hydrostatic stress are obtained from

$$S_{qa} = \frac{1}{\sqrt{2}}\sqrt{(S_{a1} - S_{a2})^2 + (S_{a2} - S_{a3})^2 + (S_{a3} - S_{a1})^2}$$

$$S_{qm} = S_{m1} + S_{m2} + S_{m3}$$

- The *S-N* approach is limited to primarily elastic and proportional loading situations.

MULTIAXIAL STRESSES (SUMMARY)

- **Strain-based** methods are based on equivalent strain. They may be suitable for inelastic loading, but their use is still limited to proportional loading.

- **Energy-based** and critical plane approaches are more general approaches and are suitable for both proportional and nonproportional multiaxial loading.

- **Critical plane** approaches reflect the physical nature of fatigue damage and can predict both fatigue life as well as the orientation of the failure plane.

- A common critical plane approach is the **Fatemi-Socie** model

$$\frac{\Delta\gamma_{max}}{2}\left(1 + k\frac{\sigma_{n,max}}{S_y}\right) = \frac{\tau'_f}{G}(2N_f)^{b_0} + \gamma'_f(2N_f)^{c_0}$$

MULTIAXIAL STRESSES (SUMMARY)

- Fracture mechanics models are used to characterize the growth of macrocracks in multiaxial fatigue, where **mixed mode** crack growth often exists.

- **Equivalent stress intensity factors** have been used to relate mixed mode crack growth data to mode I data from uniaxial loading.

- **Notches and variable amplitude loading** introduce additional complexities, particularly for nonproportional loading. Several methods to incorporate these effects have been developed.

MULTIAXIAL STRESSES
(DOS AND DON'TS IN DESIGN)

- Don't ignore the presence of multiaxial stress states as they can significantly affect fatigue behavior. The state of stress at the root of a notch is usually multiaxial, even under uniaxial loading.

- Do check whether the alternating stresses or strains have fixed principal directions. If so, the loading is proportional and fairly simple methods for life estimation can be used.

- Don't ignore the effects of nonproportional cyclic loading since it can produce additional cyclic hardening and often results in a shorter fatigue life compared to proportional loading.

MULTIAXIAL STRESSES
(DOS AND DON'TS IN DESIGN)

- Do recognize that application of equivalent stress and equivalent strain approaches to multiaxial fatigue are limited to simple proportional or in-phase loading. Other methods such as the critical plane approach are more suitable for more complex nonproportional or out-of-phase loading.

- Do check whether the fatigue damage mechanism for the given material and loading is dominated by shear or by tensile cracking. Different fatigue damage models apply to each case.