# The Fracture Mechanics Method (da/dN-ΔK)

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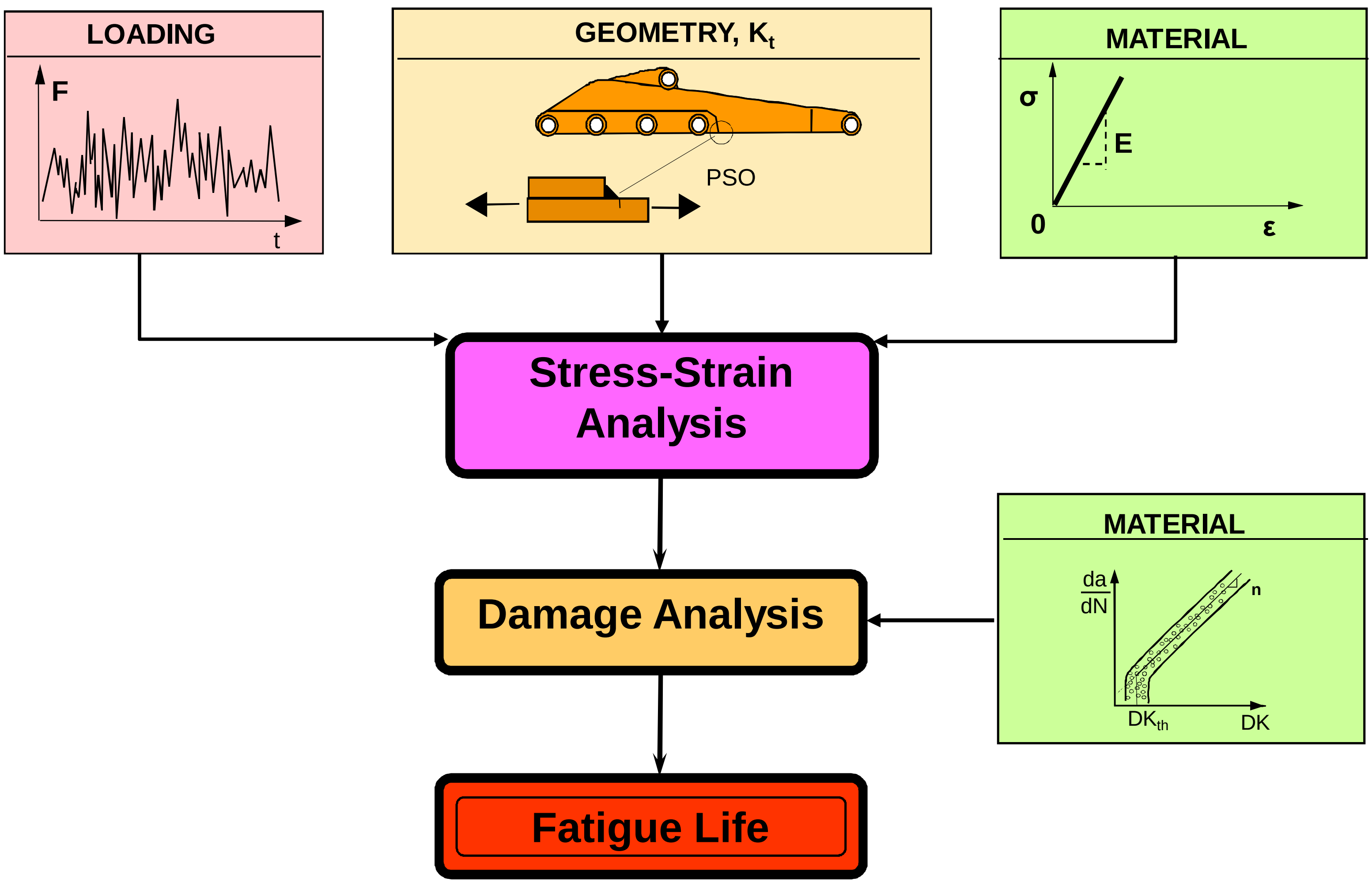
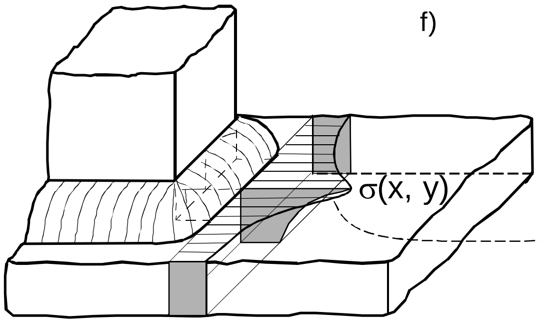
# Information path for fatigue life estimation based on the da/dN-ΔK method



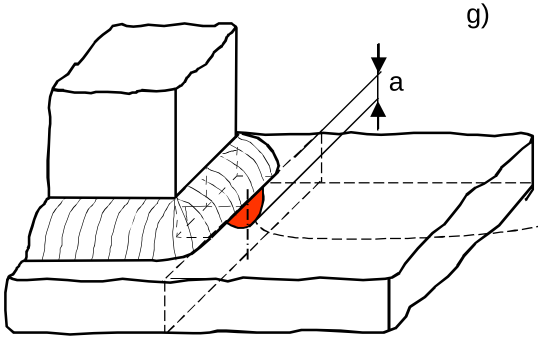

 

# Steps in Fatigue Life Prediction Procedure Based on the da/dN-ΔK Approach *(cont'd)*

f)




g)




**Stress intensity factor, K (indirect method)**

Weight function, m(x,y)

$$K = \iint_A \sigma(x,y)\, m(x,y)\, dxdy$$

$$Y = \frac{K}{\sigma_n \sqrt{\pi a}}$$

**Stress intensity factor, K (direct method)**

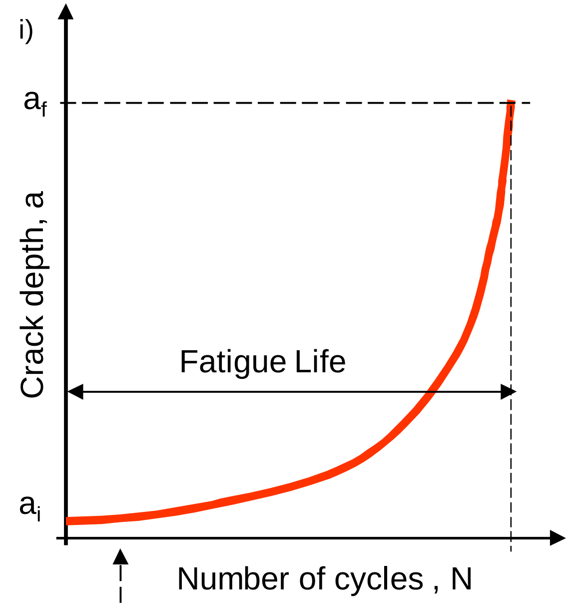
$$K_I = \sigma_{yFE} \sqrt{2\pi x_{FE}}$$

*or*

$$K = \sqrt{E \frac{dU}{da}} = \sqrt{EG}$$

$$Y = \frac{K}{\sigma_n \sqrt{\pi a}}$$

i)




h)

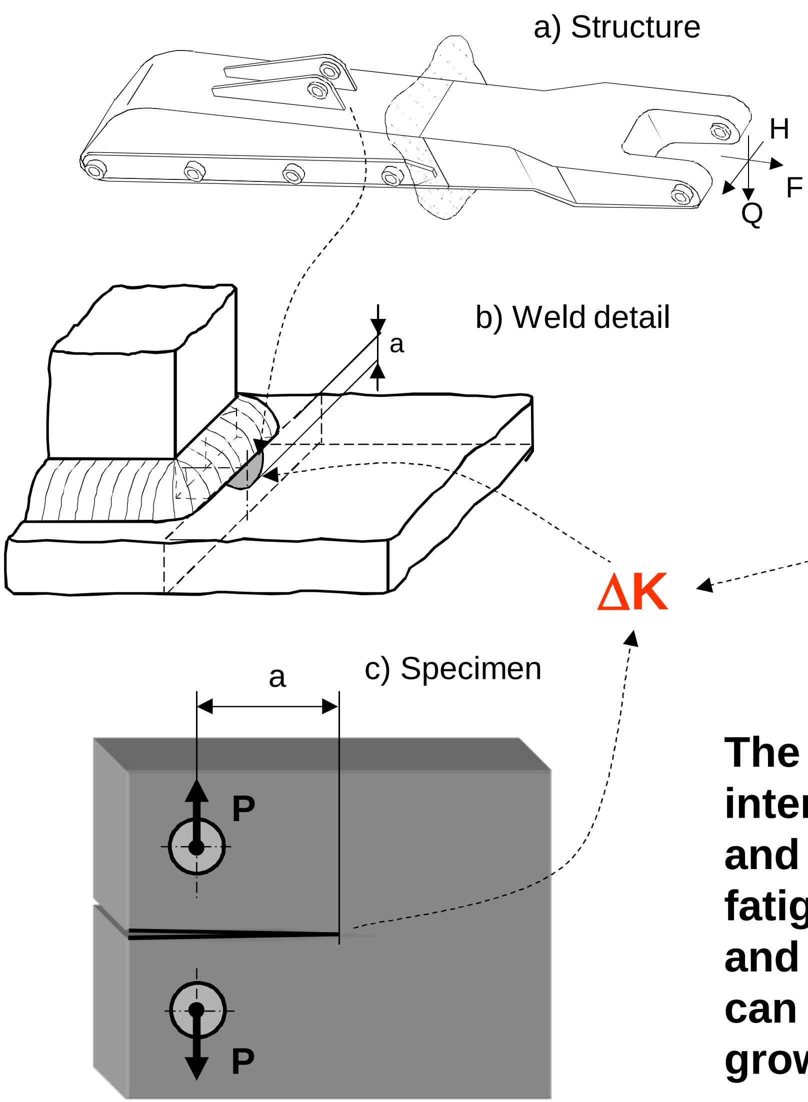
**Integration of Paris' equation**

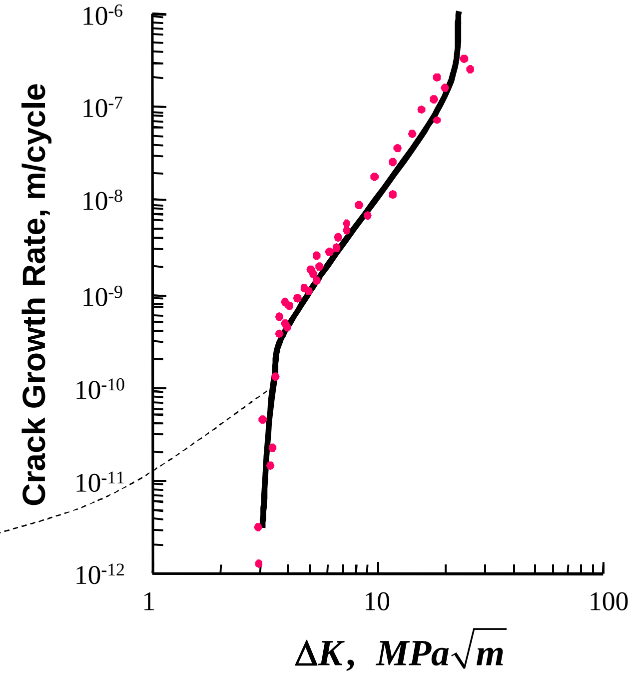
$$\Delta a_i = C\left(\Delta K_i\right)^m \Delta N_i$$

$$a_f = a_0 + \sum_{i=1}^{N} \Delta a_i$$

$$N = \sum \Delta N_i$$

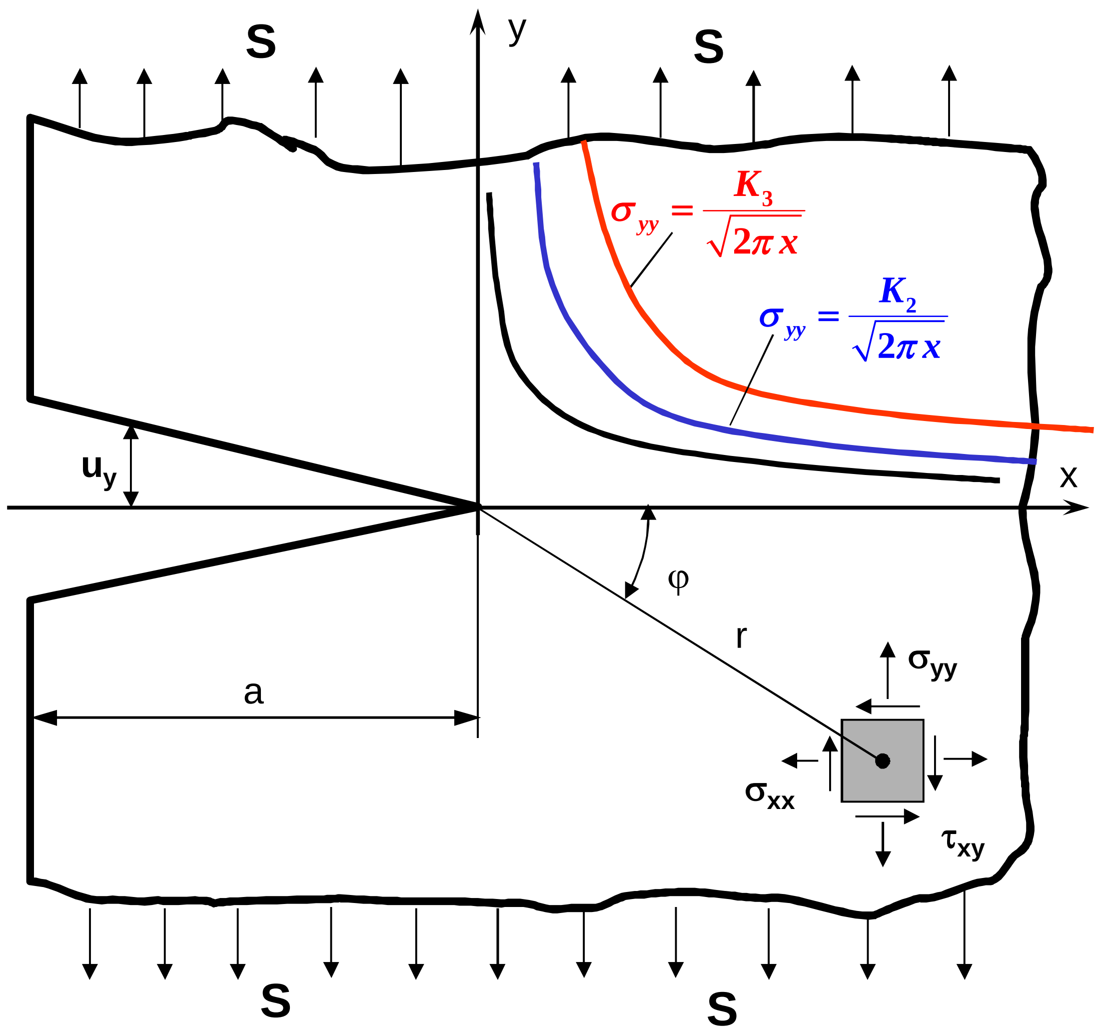
# The Similitude Concept in the da/dN – ΔK Method




**The Similitude Concept states that if the stress intensity K for a crack in the actual component and in the test specimen are the same, then the fatigue crack growth response in the component and in the specimen will also be the same and can be described by the material fatigue crack growth curve da/dN - ΔK.**

# Crack tip stress dependence on the stress intensity factor K




$K_1 < K_2 < K_3$

**Stress components, $\sigma_{ij}$, at the crack tip depend on the stress intensity factors $K_I$ which is influenced by:**

**- the load, S**

**-crack dimensions, a**

**-geometry, Y**

**The stress field , $\sigma_{ij}$, around the crack tip can be described by one universal function valid for all cracks of Mode I, i.e. for φ=0**

$$\sigma_{yy} = \frac{K}{\sqrt{2\pi x}}$$

# G. Irwin's fundamental Fracture Mechanics principles:

**1. The near crack tip stress field expressions above are universal, i.e. the stress distributions in the vicinity of the crack tip have the same general mathematical form regardless of the crack geometry, loading and geometrical shape of the body.**

**2. The strain energy release rate $G_I$ is related to the stress intensity factor $K_I$ and therefore it is justified (and easier) to calculate the strain energy release rate (and the critical stress) from the purely elastic local (near the crack tip) stress distribution (i.e. from the Stress Intensity Factor).**

$$G_I = \frac{\left(S\sqrt{\pi a}Y\right)^2}{E} = \frac{K_I^2}{E} \; - \; plane \;\; stress$$

$$G_I = \frac{\left(S\sqrt{\pi a}Y\right)^2}{E}\left(1-\nu^2\right) = \frac{K_I^2}{E}\left(1-\nu^2\right) \; - \; plane \;\; strain$$

**A crack becomes unstable (fracture) when the stress intensity factor, $K_I$, exceeds the critical, for given material, stress intensity factor $K_{Ic}$!**

$$K_I > K_{Ic}$$

**Strength parameters traditionally used for the sttrength analysis of engineering components and structures:**

| | Force [MN] | Stress [MPa] | Stress Intensity Factor [MPa√m] |
|---|---|---|---|
| | $P$ | $\sigma$ | $K_I$ |
| | $P_Y$ | $\sigma_Y$ | $K_c$ |
| | $P_{cr}$ | $\sigma_{uts}$ | $K_{Ic}$ |
| | Leonardo da Vinci<br>17th-century | Euler, Cauchy<br>19th -century | Irwin<br>20th-century |

**Fracture Mechanics parameters used for the strength analysis of engineering components and structures:**

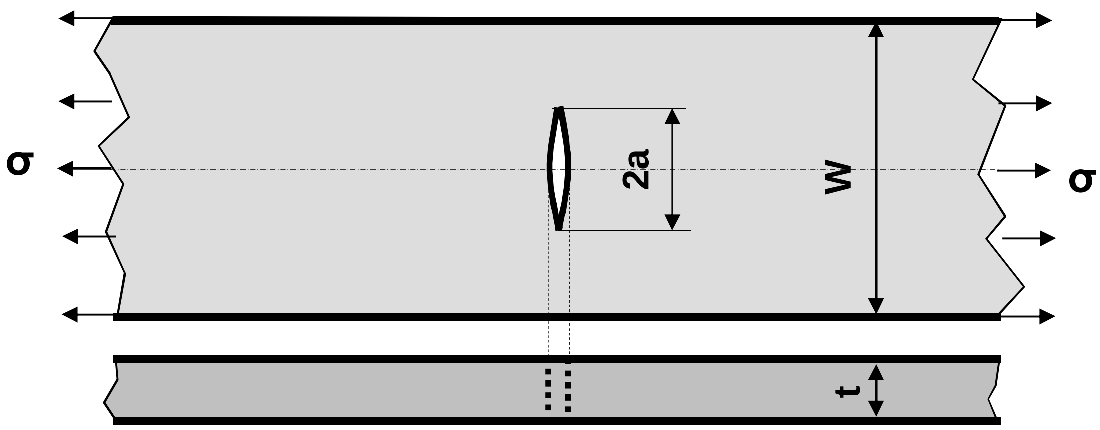
# General Stress Intensity Factor Expressions for Cracks in Mode I

**The stress intensity factor is defined as:**

$$K_I = S\sqrt{\pi a} \cdot Y$$

**in which S is the stress (usually the nominal) away from the crack. The geometry factor, Y , accounts for the effect of geometry of the crack and the body, the boundary conditions and the type of loading.**
**Determining stress intensity factor means in essence the derivation of the function describing the geometrical factor Y. One of the confusing issues while determining stress intensity factors is that the remotely applied stress S and the geometry factor Y are inter-related. The value of parameter Y depends on the definition of the remote (termed often as nominal) stress S. In cases where the nominal or hot spot stress is well defined there is no problem in the definition of the remote stress S. However, if the stress distribution is non-uniform it may not be clear which stress should be used in the expression for the stress intensity factor. Theoretically, any reference stress S can be chosen for the determination of the geometrical factor Y, as far as the stress varies proportionally with the applied load. However, the user of given expression for K has to use the same definition of the reference stress while carrying out fatigue and fracture analyses. Nominal or the maximum stress in the case of non-uniform stress distributions is most often used in stress intensity factor expressions. Therefore, it is a good professional practice to define the reference stress S when quoting the geometry factor Y.**

# Center Crack Plate under Uniform Tension




(1)
$$K_I = \sigma\sqrt{\pi a}\cdot F_I(\alpha),\ \ \alpha = \frac{2a}{W},\ \ Y = F_I(\alpha)$$

$$F_I(\alpha) = \sqrt{\sec\left(\frac{\alpha\cdot\pi}{2}\right)}$$

*Reference: C.F. Federsen (1), H. Tada (2)*

*Method: Empirical formula based on Isida's results*

*Accuracy: +0.3% for 2a/W≤0.7 and 1.0% for 2a/W=0.8*

**or**

(2) $$K_I = \sigma\sqrt{\pi a}\cdot F(\alpha)\cdot\left(1 - 0.025\alpha^2 + 0.06\alpha^4\right)$$

*Accuracy: Better than 0.2% for any value of α*

*(Y. Murakami et. al)*

# The SIF geometry correction factor Y; $K_I = S\sqrt{\pi a}\cdot Y$; (central crack)

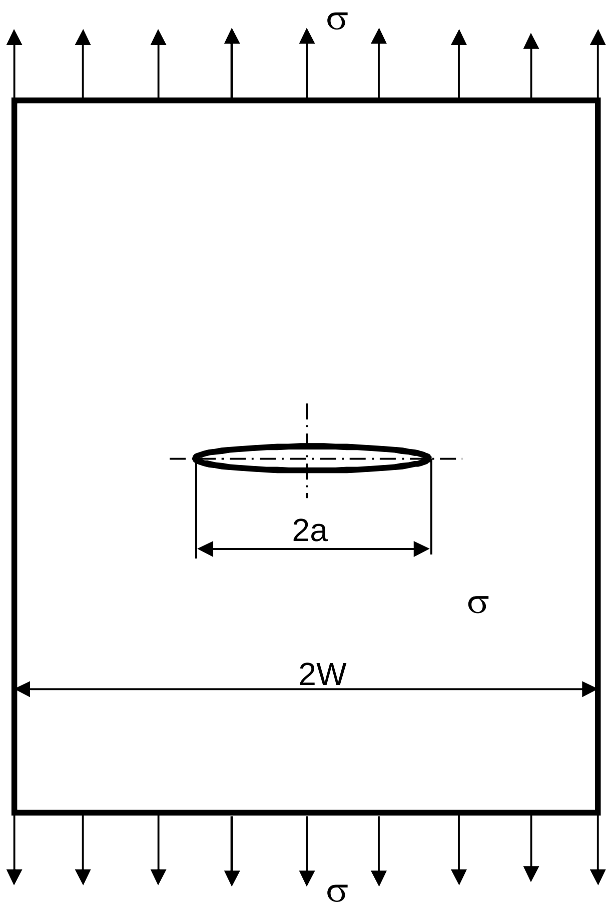
| | Geometry correction factor, $F_I(\alpha) = Y$ | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **2a/W** | 0.000 | 0.001 | 0.002 | 0.003 | 0.004 | 0.005 | 0.006 | 0.007 | 0.008 | 0.009 | 0.010 |
| 0.200 | 1.025 | 1.026 | 1.026 | 1.026 | 1.026 | 1.027 | 1.027 | 1.027 | 1.028 | 1.028 | 1.058 |
| 0.210 | 1.028 | 1.028 | 1.029 | 1.029 | 1.029 | 1.029 | 1.030 | 1.030 | 1.030 | 1.031 | 1.031 |
| 0.220 | 1.031 | 1.031 | 1.032 | 1.032 | 1.032 | 1.032 | 1.033 | 1.033 | 1.033 | 1.034 | 1.034 |
| 0.230 | 1.034 | 1.034 | 1.035 | 1.035 | 1.035 | 1.035 | 1.036 | 1.036 | 1.036 | 1.037 | 1.037 |
| 0.240 | 1.037 | 1.037 | 1.038 | 1.038 | 1.038 | 1.039 | 1.039 | 1.039 | 1.040 | 1.040 | 1.040 |
| 0.250 | 1.040 | 1.041 | 1.041 | 1.041 | 1.042 | 1.042 | 1.042 | 1.043 | 1.043 | 1.043 | 1.044 |
| 0.260 | 1.044 | 1.044 | 1.045 | 1.045 | 1.045 | 1.046 | 1.046 | 1.046 | 1.047 | 1.047 | 1.047 |
| 0.270 | 1.047 | 1.048 | 1.048 | 1.049 | 1.049 | 1.049 | 1.050 | 1.050 | 1.051 | 1.051 | 1.051 |
| 0.280 | 1.051 | 1.052 | 1.052 | 1.052 | 1.053 | 1.053 | 1.054 | 1.054 | 1.054 | 1.055 | 1.055 |
| 0.290 | 1.055 | 1.056 | 1.056 | 1.056 | 1.057 | 1.057 | 1.058 | 1.058 | 1.059 | 1.059 | 1.059 |
| 0.300 | 1.059 | 1.060 | 1.060 | 1.061 | 1.062 | 1.062 | 1.062 | 1.062 | 1.063 | 1.063 | 1.064 |
| 0.310 | 1.064 | 1.064 | 1.064 | 1.067 | 1.066 | 1.066 | 1.066 | 1.067 | 1.067 | 1.068 | 1.068 |
| 0.320 | 1.068 | 1.069 | 1.069 | 1.070 | 1.070 | 1.071 | 1.071 | 1.072 | 1.072 | 1.072 | 1.073 |
| 0.330 | 1.073 | 1.073 | 1.073 | 1.074 | 1.075 | 1.075 | 1.076 | 1.076 | 1.077 | 1.077 | 1.078 |
| 0.340 | 1.078 | 1.078 | 1.078 | 1.079 | 1.080 | 1.080 | 1.081 | 1.081 | 1.082 | 1.082 | 1.083 |
| 0.350 | 1.083 | 1.083 | 1.083 | 1.085 | 1.085 | 1.086 | 1.086 | 1.087 | 1.087 | 1.088 | 1.088 |
| 0.360 | 1.088 | 1.089 | 1.089 | 1.090 | 1.090 | 1.091 | 1.092 | 1.092 | 1.093 | 1.093 | 1.094 |
| 0.370 | 1.094 | 1.094 | 1.094 | 1.096 | 1.096 | 1.097 | 1.097 | 1.098 | 1.098 | 1.099 | 1.100 |
| 0.380 | 1.100 | 1.100 | 1.100 | 1.101 | 1.102 | 1.103 | 1.103 | 1.104 | 1.104 | 1.105 | 1.106 |
| 0.390 | 1.106 | 1.106 | 1.107 | 1.107 | 1.108 | 1.109 | 1.109 | 1.110 | 1.111 | 1.111 | 1.112 |
| 0.400 | 1.112 | 1.112 | 1.113 | 1.114 | 1.114 | 1.115 | 1.116 | 1.116 | 1.117 | 1.118 | 1.118 |
| 0.410 | 1.118 | 1.119 | 1.120 | 1.120 | 1.121 | 1.122 | 1.122 | 1.123 | 1.124 | 1.124 | 1.125 |
| 0.420 | 1.125 | 1.126 | 1.126 | 1.127 | 1.128 | 1.128 | 1.129 | 1.130 | 1.131 | 1.131 | 1.132 |
| 0.430 | 1.132 | 1.133 | 1.133 | 1.134 | 1.135 | 1.136 | 1.136 | 1.137 | 1.138 | 1.138 | 1.139 |
| 0.440 | 1.139 | 1.140 | 1.141 | 1.141 | 1.142 | 1.143 | 1.144 | 1.144 | 1.145 | 1.146 | 1.147 |
| 0.450 | 1.147 | 1.148 | 1.148 | 1.149 | 1.150 | 1.151 | 1.151 | 1.152 | 1.153 | 1.154 | 1.155 |
| 0.460 | 1.155 | 1.155 | 1.156 | 1.157 | 1.158 | 1.159 | 1.159 | 1.160 | 1.161 | 1.162 | 1.163 |
| 0.470 | 1.163 | 1.164 | 1.164 | 1.165 | 1.166 | 1.167 | 1.168 | 1.169 | 1.170 | 1.170 | 1.171 |
| 0.480 | 1.171 | 1.172 | 1.173 | 1.174 | 1.175 | 1.176 | 1.176 | 1.177 | 1.178 | 1.179 | 1.180 |
| 0.490 | 1.180 | 1.181 | 1.182 | 1.183 | 1.184 | 1.185 | 1.186 | 1.186 | 1.187 | 1.188 | 1.189 |
| 0.580 | 1.277 | 1.279 | 1.280 | 1.281 | 1.283 | 1.284 | 1.285 | 1.287 | 1.288 | 1.289 | 1.291 |
| 0.590 | 1.291 | 1.292 | 1.293 | 1.295 | 1.296 | 1.297 | 1.299 | 1.300 | 1.302 | 1.303 | 1.304 |
| 0.600 | 1.304 | 1.306 | 1.307 | 1.309 | 1.310 | 1.311 | 1.313 | 1.314 | 1.316 | 1.317 | 1.319 |
| 0.610 | 1.319 | 1.200 | 1.322 | 1.323 | 1.325 | 1.326 | 1.328 | 1.329 | 1.331 | 1.332 | 1.334 |
| 0.620 | 1.334 | 1.335 | 1.337 | 1.338 | 1.340 | 1.342 | 1.343 | 1.345 | 1.346 | 1.348 | 1.350 |
| 0.630 | 1.350 | 1.351 | 1.353 | 1.354 | 1.356 | 1.358 | 1.359 | 1.361 | 1.363 | 1.364 | 1.366 |
| 0.640 | 1.366 | 1.368 | 1.370 | 1.371 | 1.373 | 1.375 | 1.376 | 1.378 | 1.380 | 1.382 | 1.383 |
| 0.650 | 1.383 | 1.385 | 1.387 | 1.389 | 1.391 | 1.392 | 1.394 | 1.396 | 1.398 | 1.400 | 1.402 |

**Example**

A thick center-cracked plate of a high strength aluminum alloy is 200 mm wide and contains a crack of length 80 mm. If it fails at an applied stresses of 100 MPa, what is the fracture toughness of the alloy? What value of applied stress would produce fracture for the same length of crack in:

a) an infinite plate

b) a 120 mm wide plate?



$a)\,Finite\ width\ plate$

$2W = 200\,mm,\ 2a = 80\,mm, \sigma = 100\,MPa$

$K = \sigma\sqrt{\pi a}\cdot Y;\ \ Y = f\left(\frac{a}{W}\right);$

$See\ notation\ in\ the\ Handbook:$

$$\left(\frac{2a}{W}\right)_{handbook} = \left(\frac{2a}{2W}\right)_{example} = \left(\frac{a}{W}\right)_{example}$$

$$\left(\frac{2a}{W}\right)_{handbook} = \left(\frac{a}{W}\right)_{example} = \frac{40}{100} = 0.4$$

$$Y\left(\frac{2a}{W} = 0.4\right) = 1.112$$

$$K = \sigma\sqrt{\pi a}\cdot Y = 100\sqrt{\pi \times 0.04}\times 1.112 = 39.42\ MPa\sqrt{m}$$

$$K = K_c = 39.42\,MPa\sqrt{m}$$

$b)\,Ininitely\ wide\ plate$

$$K = \sigma\sqrt{\pi a}\cdot Y;\quad and\quad Y = 1$$

$$K_c = K;\quad 39.42 = \sigma\sqrt{\pi \times 0.04}$$

$$\sigma = \frac{39.42}{\sqrt{\pi \times 0.04}} = 111.20\,MPa$$

$c)\,Plate\ 120\,mm\ wide$

$$\left(\frac{2a}{W}\right)_{handbook} = \left(\frac{a}{W}\right)_{example} = \frac{40}{60} = 0.6666$$

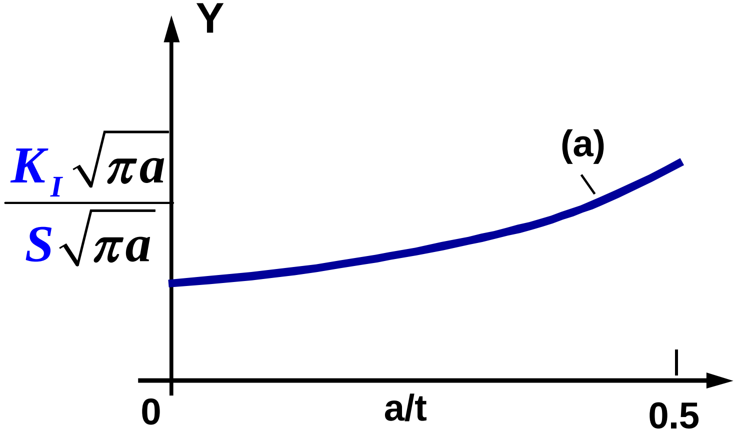
$$Y\left(\frac{2a}{W} = 0.6666\right) = 1.413$$

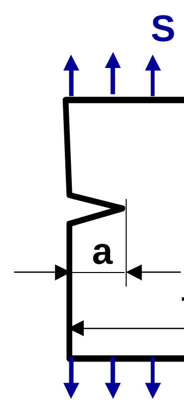
$$K_c = K;\quad 39.42 = \sigma\sqrt{\pi \times 0.04}\times 1.413$$

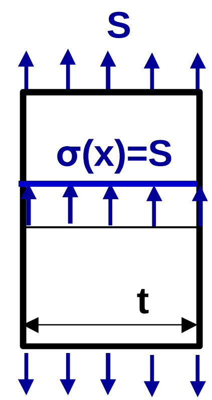
$$\sigma = \frac{39.42}{\sqrt{\pi \times 0.04}\times 1.413} = 78.7\,MPa$$

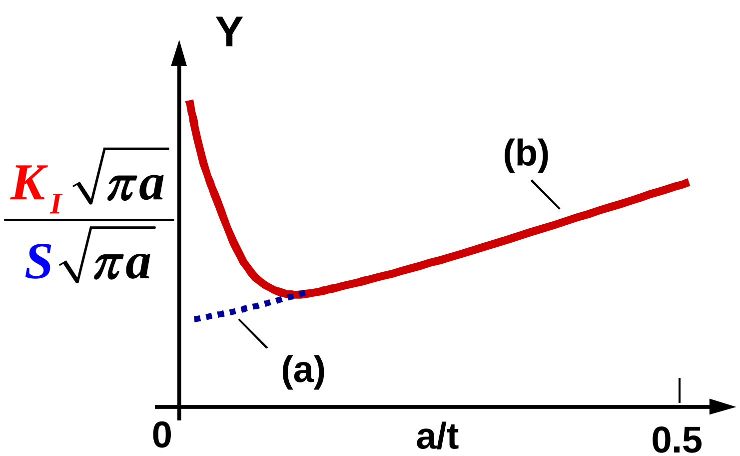
# Geometry Effects on the Stress Intensity Factor

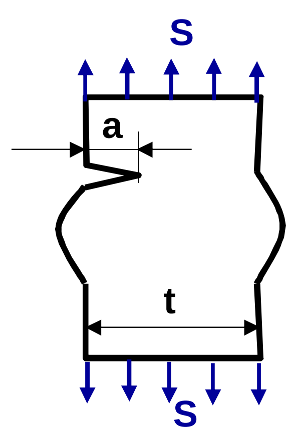
## Stress Intensity factors for cracks in a butt weldment and flat plate of the same thickness

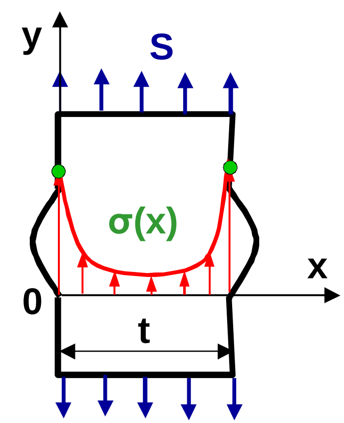


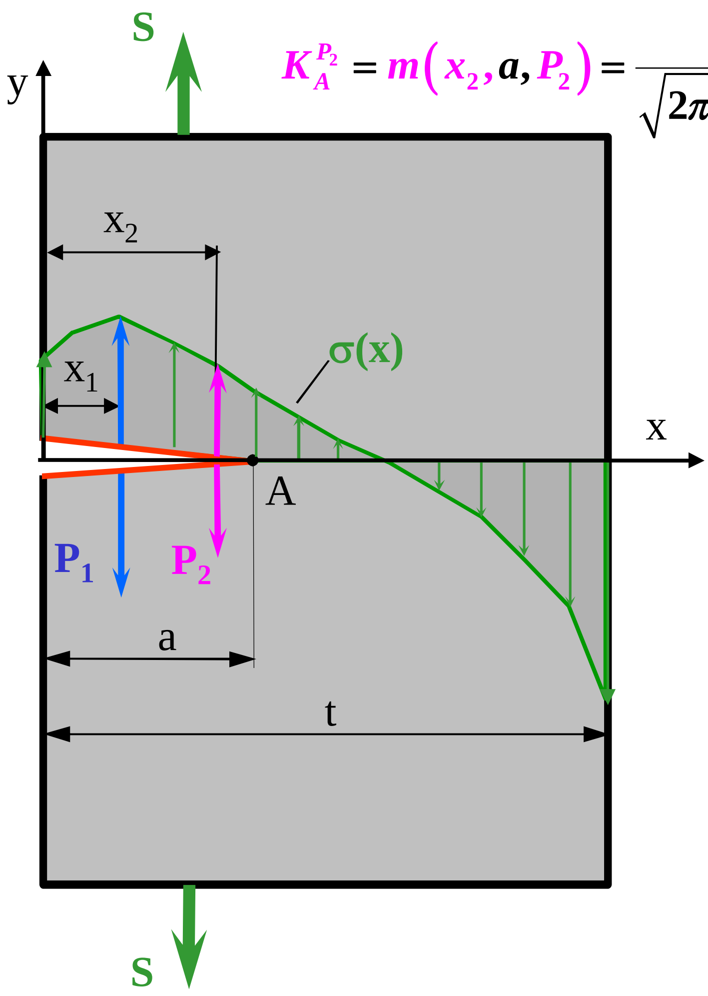



# The Weight Function method for calculating Stress Intensity Factors

$$K_A^{P_1} = m(x_1, a, P_1) = \frac{2P_1}{\sqrt{2\pi(a - x_1)}}\left[1 + M_1\left(1 - \frac{x_1}{a}\right)^{\frac{1}{2}} + M_2\left(1 - \frac{x_1}{a}\right)^{1} + M_3\left(1 - \frac{x_1}{a}\right)^{\frac{3}{2}}\right]$$

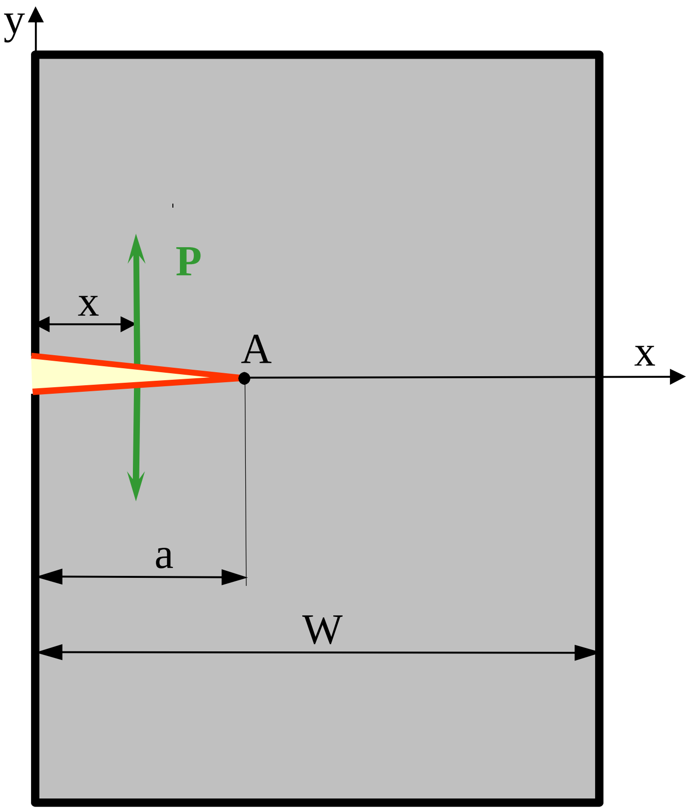
$$K_A^{P_2} = m(x_2, a, P_2) = \frac{2P_2}{\sqrt{2\pi(a - x_2)}}\left[1 + M_1\left(1 - \frac{x_2}{a}\right)^{\frac{1}{2}} + M_2\left(1 - \frac{x_2}{a}\right)^{1} + M_3\left(1 - \frac{x_2}{a}\right)^{\frac{3}{2}}\right]$$

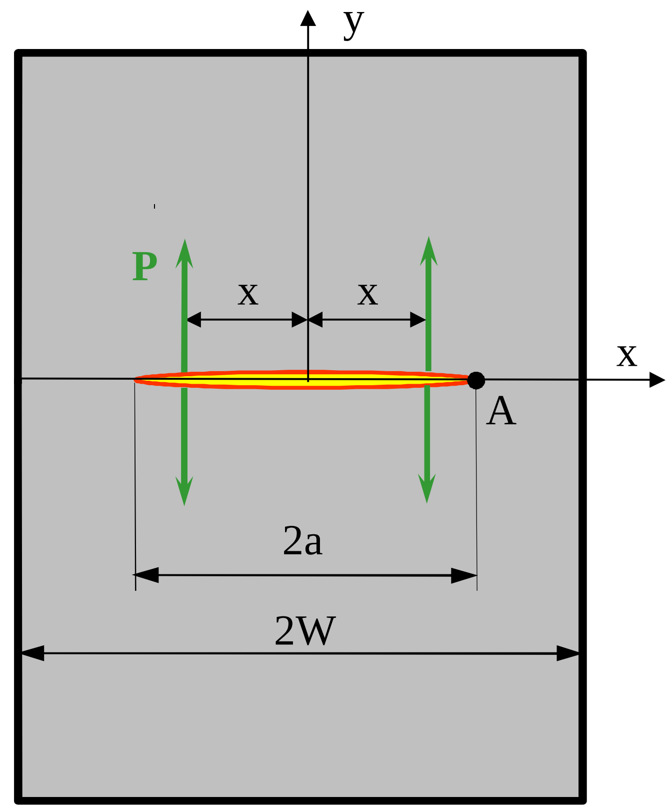



$$K_A^{P_1+P_2} = m(x_1, a, P_1) + m(x_2, a, P_2)$$

$$K_A^{\sigma(x)} = \int_0^a [\sigma(x)\, m(x, a)]\, dx$$

# Geometrical parameters and notation for weight functions




$$K_A^{P_1} = m(x,a,P) = \frac{2}{\sqrt{2\pi(a-x)}}\left[1 + M_1\left(1-\frac{x}{a}\right)^{\frac{1}{2}} + M_2\left(1-\frac{x}{a}\right)^{1} + M_3\left(1-\frac{x}{a}\right)^{\frac{3}{2}}\right]$$

## Central through crack in a finite width plate subjected to symmetric loading

$$M_1 = 0.06987 + 0.40117\left(\frac{a}{w}\right) - 5.5407\left(\frac{a}{w}\right)^2 + 50.0886\left(\frac{a}{w}\right)^3 - 200.699\left(\frac{a}{w}\right)^4 + 395.552\left(\frac{a}{w}\right)^5 - 377.939\left(\frac{a}{w}\right)^6 + 140.218\left(\frac{a}{w}\right)^7$$

$$M_2 = -0.09049 - 2.14886\left(\frac{a}{w}\right) + 22.5325\left(\frac{a}{w}\right)^2 - 89.6553\left(\frac{a}{w}\right)^3 + 210.599\left(\frac{a}{w}\right)^4 - 239.445\left(\frac{a}{w}\right)^5 + 111.128\left(\frac{a}{w}\right)^6$$

$$M_3 = 0.427216 + 2.56001\left(\frac{a}{w}\right) - 29.6349\left(\frac{a}{w}\right)^2 + 138.40\left(\frac{a}{w}\right)^3 - 347.255\left(\frac{a}{w}\right)^4 + 457.128\left(\frac{a}{w}\right)^5 - 295.882\left(\frac{a}{w}\right)^6 + 68.1575\left(\frac{a}{w}\right)^7$$

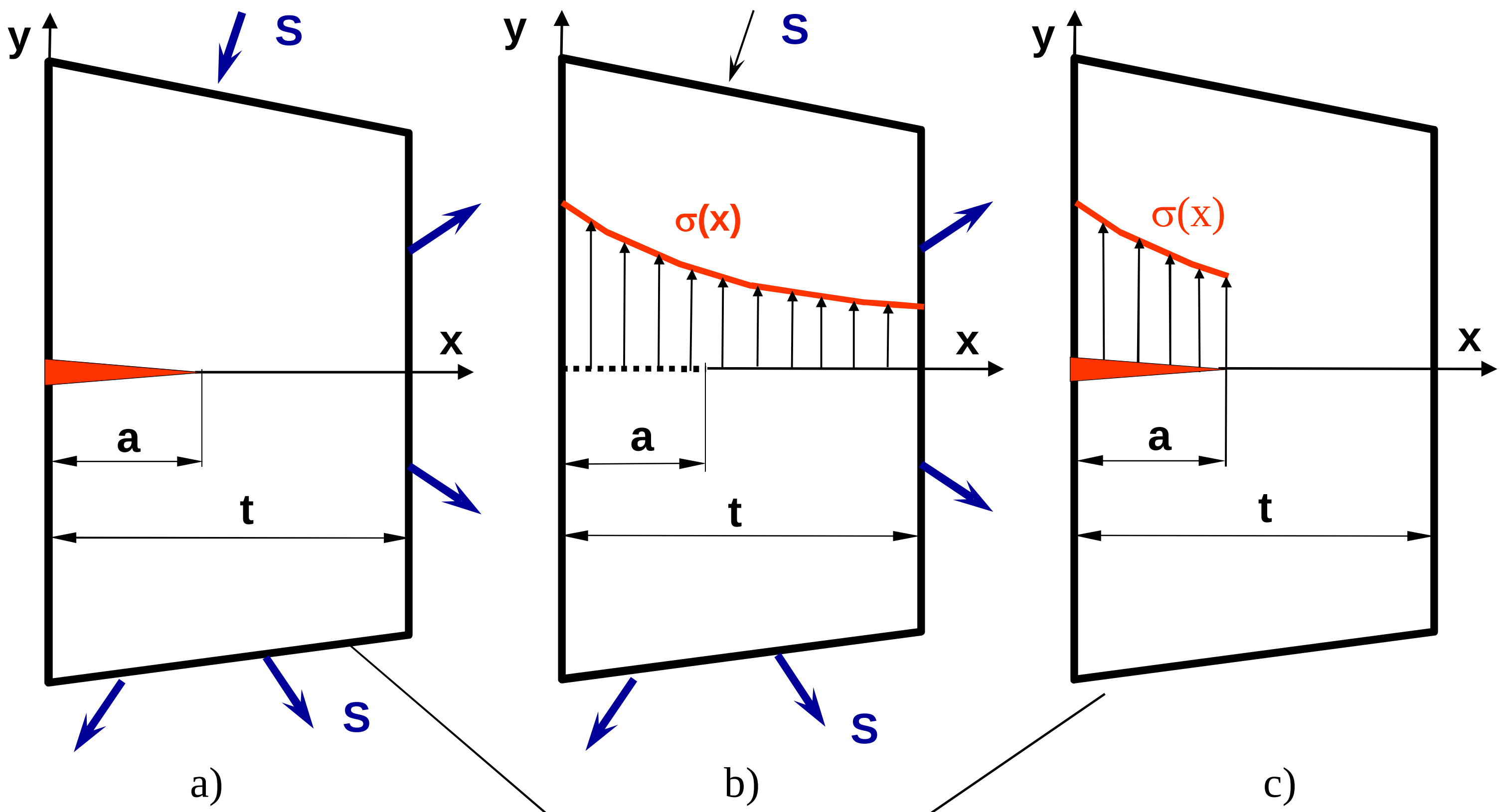
## Edge crack in a finite width plate

$$M_1 = 0.0719768 - 1.51346\left(\frac{a}{w}\right) - 61.1001\left(\frac{a}{w}\right)^2 + 1554.95\left(\frac{a}{w}\right)^3 - 14583.8\left(\frac{a}{w}\right)^4 + 71590.7\left(\frac{a}{w}\right)^5 - 205384\left(\frac{a}{w}\right)^6 + 356469\left(\frac{a}{w}\right)^7$$
$$- 368270\left(\frac{a}{w}\right)^8 + 208233\left(\frac{a}{w}\right)^9 - 49544\left(\frac{a}{w}\right)^{10}$$

$$M_2 = 0.246984 + 6.47543\left(\frac{a}{w}\right) + 176.457\left(\frac{a}{w}\right)^2 - 4058.76\left(\frac{a}{w}\right)^3 + 37303.8\left(\frac{a}{w}\right)^4 - 181755\left(\frac{a}{w}\right)^5 + 520551\left(\frac{a}{w}\right)^6 - 904370\left(\frac{a}{w}\right)^7$$
$$+ 936863\left(\frac{a}{w}\right)^8 - 531940\left(\frac{a}{w}\right)^9 + 127291\left(\frac{a}{w}\right)^{10}$$

$$M_3 = 0.529659 - 22.3235\left(\frac{a}{w}\right) + 532.074\left(\frac{a}{w}\right)^2 - 5479.53\left(\frac{a}{w}\right)^3 + 28592.2\left(\frac{a}{w}\right)^4 - 81388.6\left(\frac{a}{w}\right)^5 + 128746\left(\frac{a}{w}\right)^6 - 106246\left(\frac{a}{w}\right)^7$$
$$+ 35780.7\left(\frac{a}{w}\right)^8$$

# The Weight Function method for calculating Stress Intensity Factors




$$K_a = K_b$$

The Stress Intensity Factor for any loading case is equal to the stress intensity factor obtained by applying to the crack faces the stresses that used to be there when there was no crack.

# Stepwise Procedure for the Stress Intensity Calculation using the Weight Function Method

**1. Calculate stress distribution $\sigma(x)$ in the prospective crack plane in the absence of the crack (un-cracked body, linear elastic analysis).**

$$\sigma\left(x\right)=f\left(\sigma_0,x\right)$$

**2. Apply the stress distribution $\sigma(x)$ to the crack surface as tractions.**

**3. Choose appropriate weight function, i.e. parameters $M_1$, $M_2$ and $M_3$.**

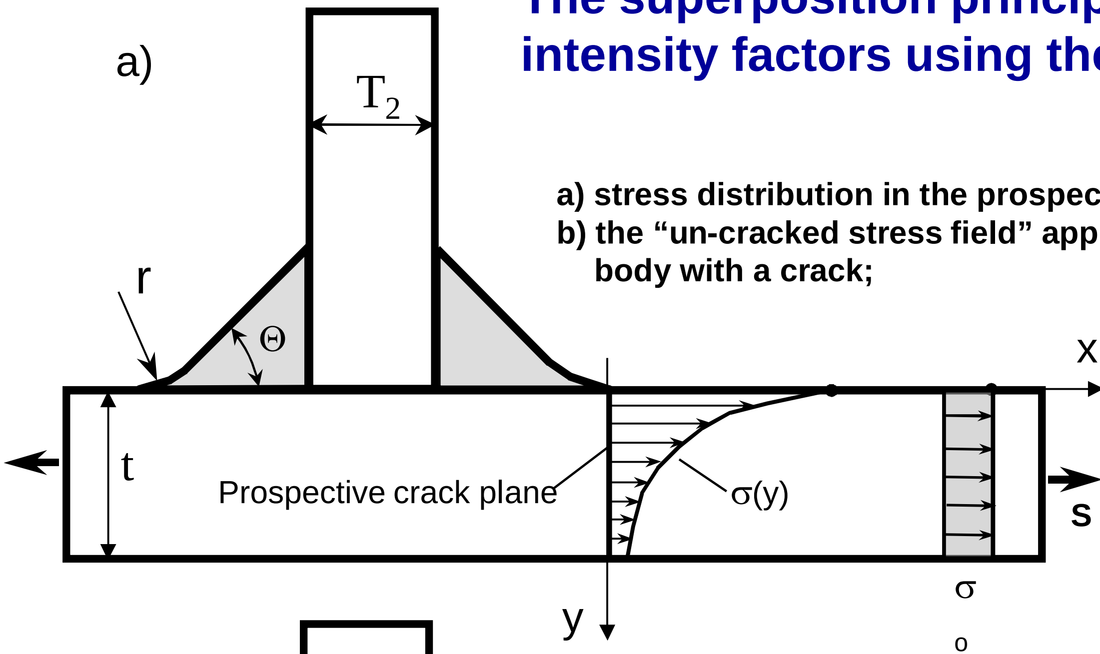
$$m(x,a)=\frac{2}{\sqrt{2\pi\left(a-x\right)}}\left[1+M_1\left(1-\frac{x}{a}\right)^{1/2}+M_2\left(1-\frac{x}{a}\right)^{1}+M_3\left(1-\frac{x}{a}\right)^{3/2}\right]$$

**4. Integrate the product of the stress distribution $\sigma(x)$ and the weight function m(x,a).**

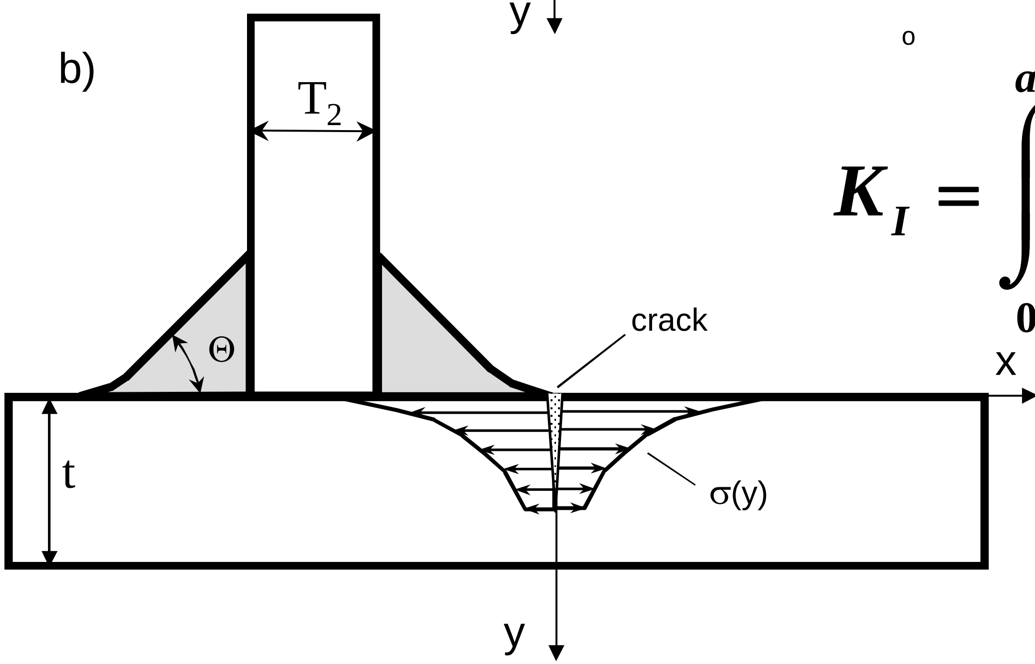
$$K=\int_{-a}^{+a}\sigma\left(x\right)m(x,a)dx$$

# The superposition principle for calculation of stress intensity factors using the weight function approach;

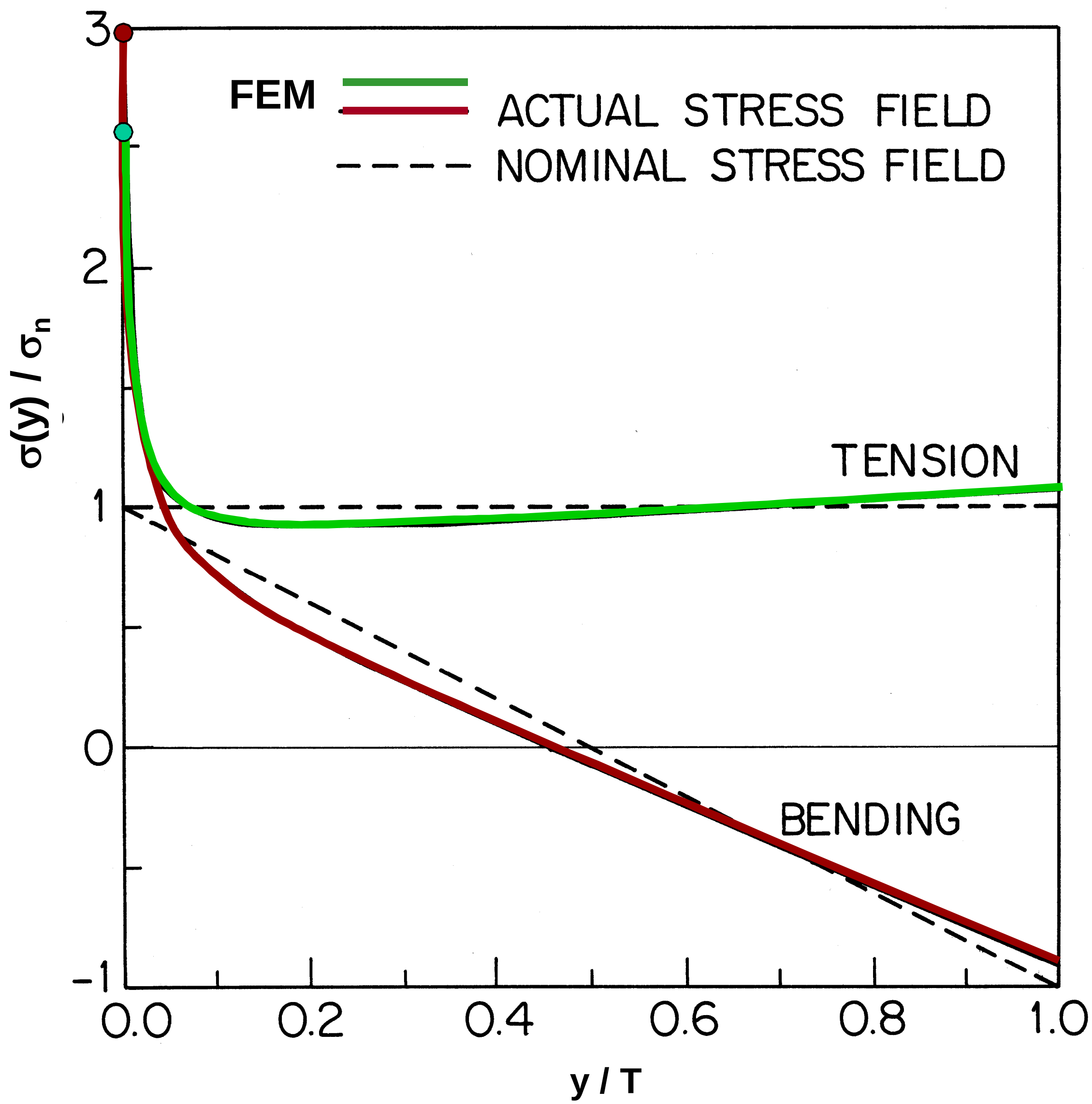
**a) stress distribution in the prospective crack plane in the un-cracked body;**
**b) the "un-cracked stress field" applied to the crack surfaces of identical body with a crack;**




$$K_I = \int_0^a m\left(y, \frac{a}{t}\right) \cdot \sigma(y) \cdot dy$$



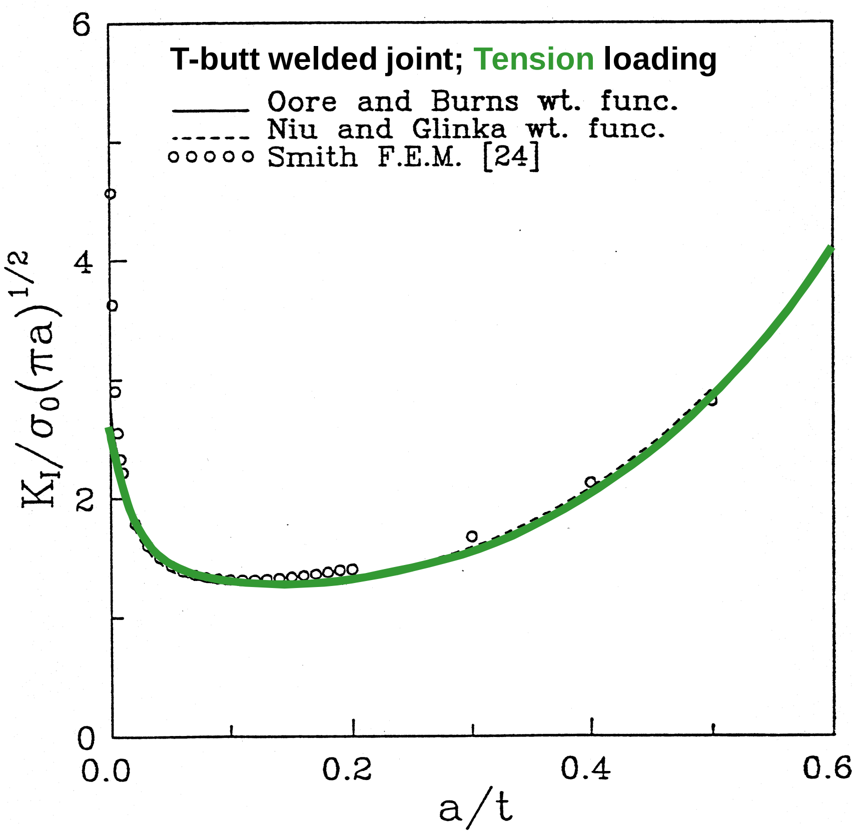
# Through the plate thickness stress distributions in a T-butt weldment obtained for r/t = 1/25, Θ = 45$^o$ (in the weld toe cross section)

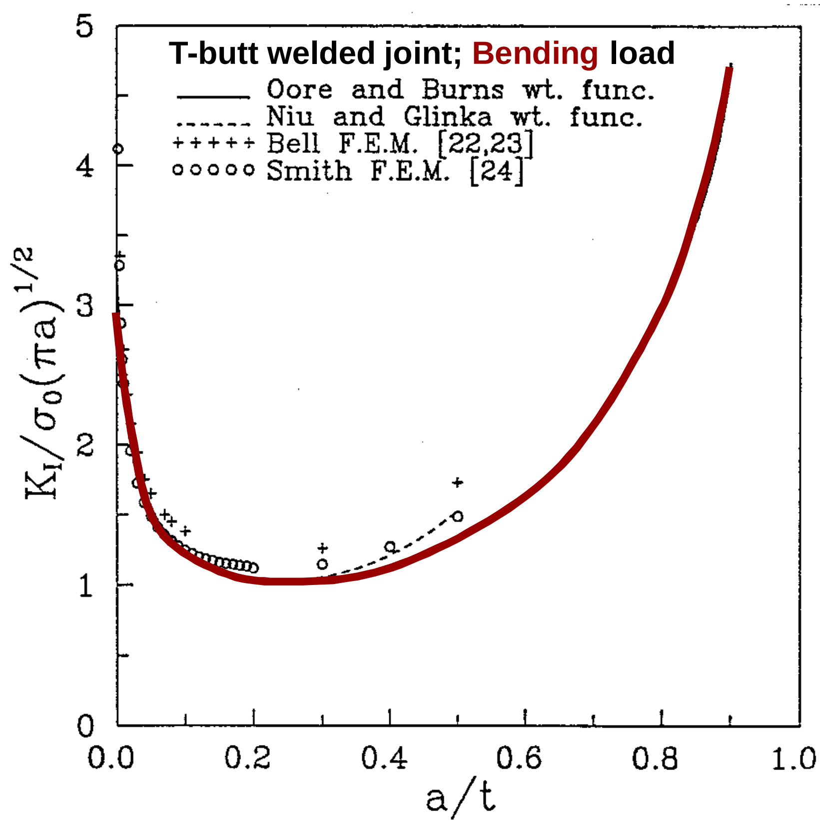


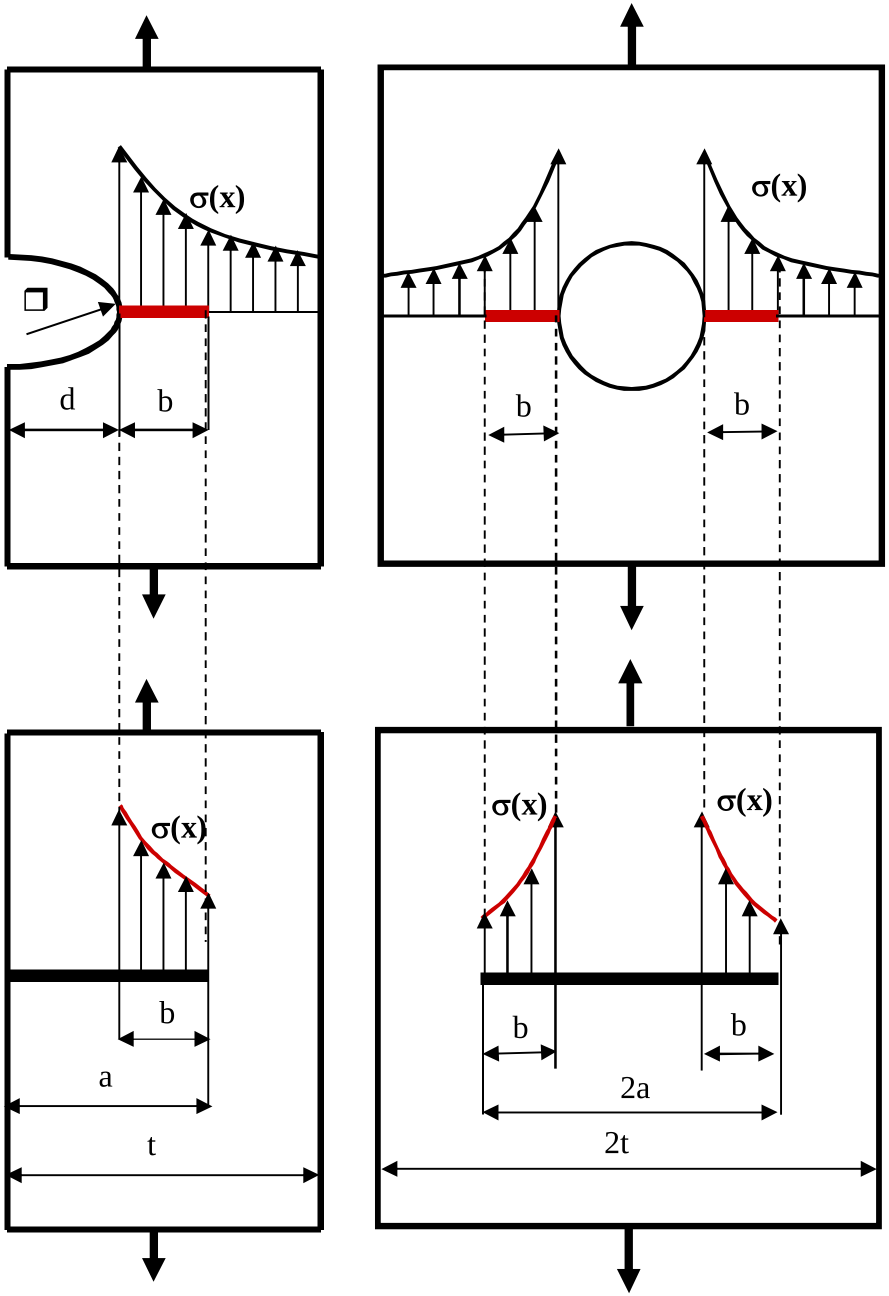

 

# Geometrical Stress Intensity Correction Factor "Y" for an Edge Crack Emanating from the Weld Toe

*(Comparison of WF and FEM data)*






## Calculation of SIF for cracks at notches using the weight functions for edge and through cracks

# Fracture Mechanics Approach to Fatigue Crack Growth Analysis

•Fatigue crack growth equations

•Integration of  fatigue crack growth expressions

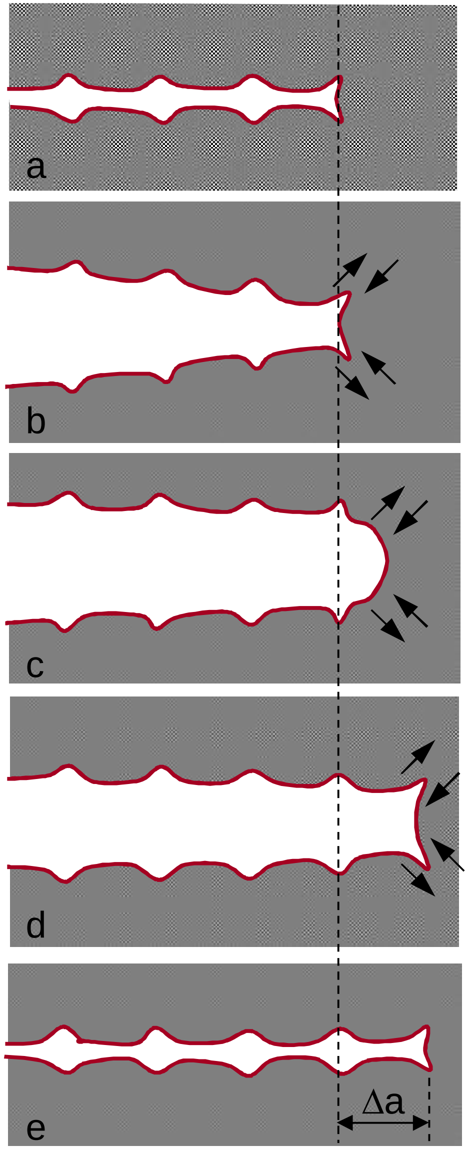
•The effect of the initial crack size

•The effect of the weld geometry

•Residual stress effect

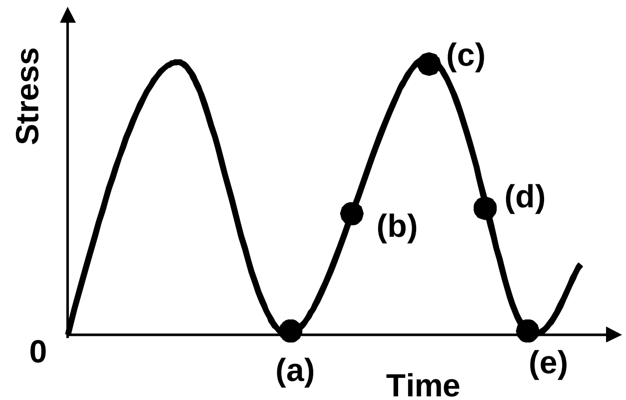
•Example

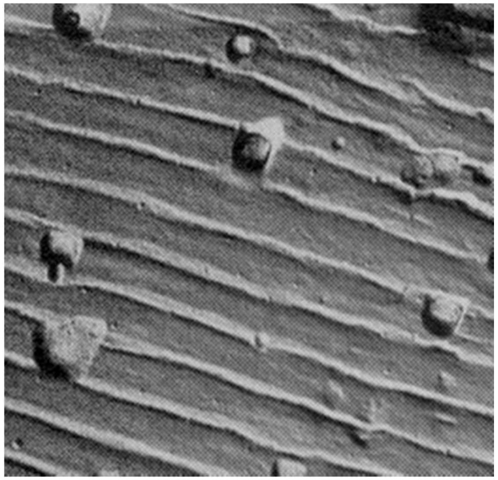
# Fatigue Crack Growth Micro-Mechanism




(C. Laird, ASTM, STP 415, 1967)

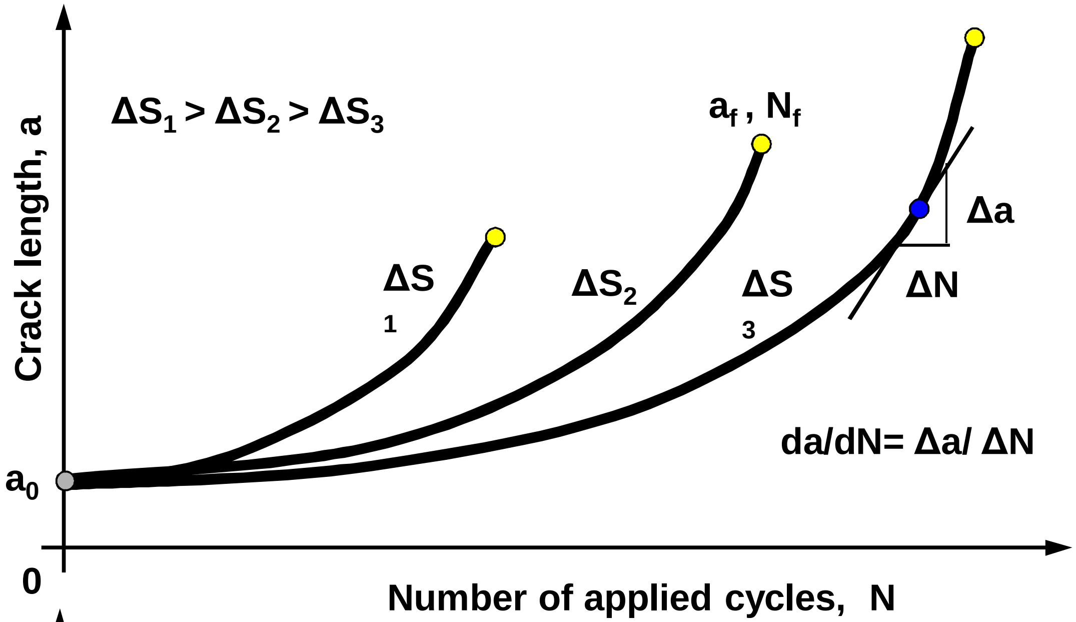
A sharp crack in a tension stress field causes a high stress concentration at the its tip resulting in slip and plastic deformation in the crack tip vicinity. The material above and below the crack tip may slip along a favorable slip plane in the direction of maximum shear stress.

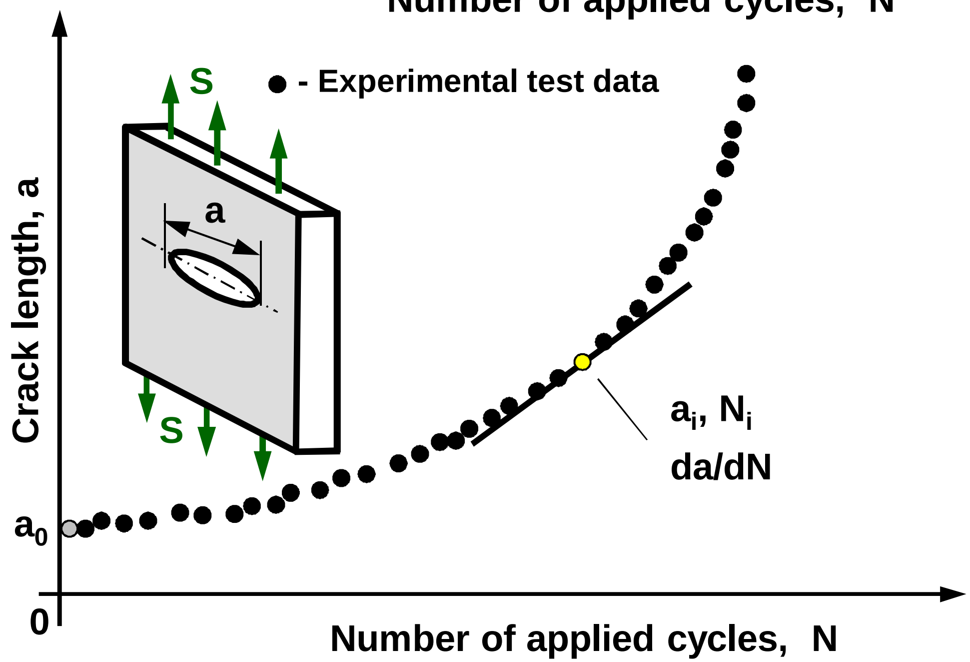


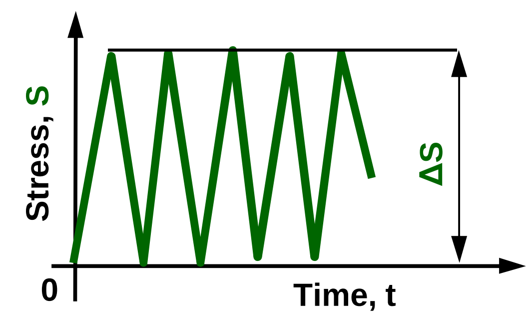
(R.Pelloux, ASTM, STP 415, 1967)

# Experimental data for the determination of the fatigue crack growth curve




**Applied nominal stress history**




**The 'a vs. N' data is obtained in practice by periodic measurement of the crack length, a, together with the number of cycles, N. The raw data is usually given in the form of series of points as shown in the figure.**

The ***Fracture Mechanics*** approach to fatigue or the da/dN - ΔK method is a technique based on the analysis of fatigue crack growth. The combination of load/stress and geometry parameters, necessary for the quantification of damage due to crack growth, is represented by the stress intensity factor, K, in the case of monotonic load and by the range of the stress intensity factor, ΔK, in the case of cyclic loading.

The fatigue material properties are characterized by the threshold stress intensity range, $\Delta K_{th}$, the fatigue crack growth rate relationship, da/dN vs. ΔK, and the critical stress intensity factor, $K_c$, to be often the same as the fracture toughness, $K_{Ic}$. The crack growth rate is then described by an expression being function of the stress intensity range:

$$\frac{da}{dN} = f\left(\Delta K\right),$$

The stress intensity range associated with a stress cycle is calculated as:

$$\Delta K = K_{\max} - K_{\min} = S_{\max}\sqrt{\pi a} \times Y - S_{\min}\sqrt{\pi a} \times Y$$

where – a is the crack size, $S_{max}$ and $S_{min}$ is the maximum and minimum nominal (or reference) stress respectively, characterizing a stress cycle, and Y is the geometry correction factor. The aim of the final analysis of the da/dN-ΔK data is to determine necessary constants and parameters appearing in expression f(ΔK).

It should be noted that the 'da/dN - ΔK' curve in fracture mechanics represents the material fatigue resistance similarly to the S-N curve in the nominal stress approach or the 'ε - N' relationship in the local strain-life methodology.

As soon as the crack growth curve for the material of interest is known the fatigue life of the structural component can be determined as shown in the figure below.

The notation for the cyclic stress history parameters and the steps necessary for the determination of the da/dN - ΔK relationship are explained later in the following sections of the notes.

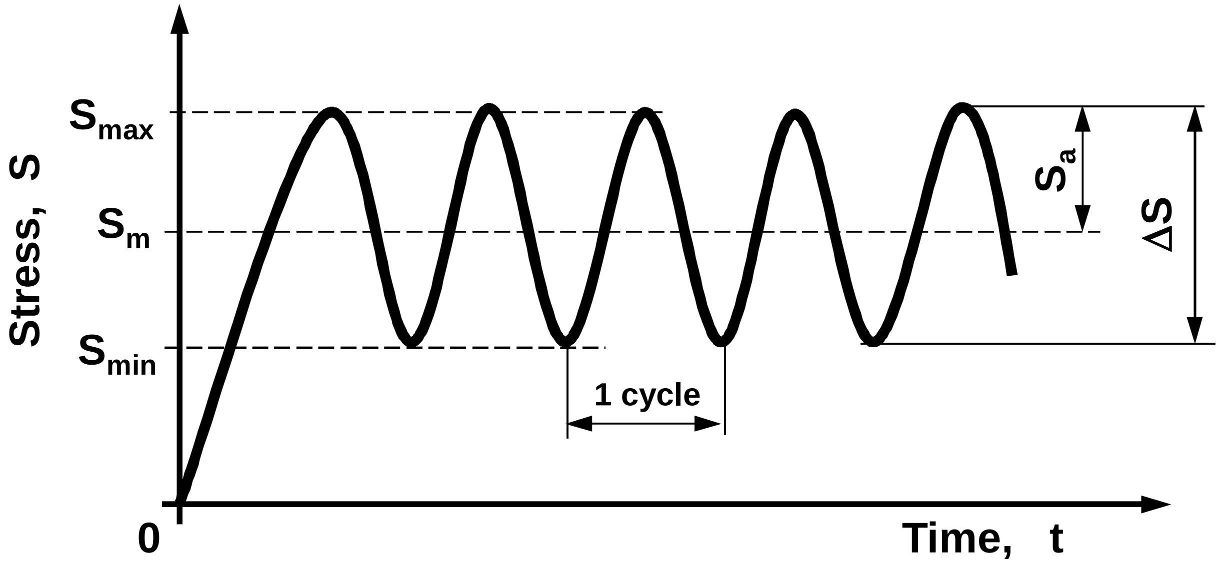
The fatigue life in terms of the number of cycles necessary to propagate the crack from its initial size, $a_0$, to the final or critical crack size, $a_f$, is determined by integrating the crack growth equation.

$$N = \int_{a_0}^{a_f} \frac{da}{f(\Delta K)} = \int_{a_0}^{a_f} \frac{da}{f\left(\Delta S\sqrt{\pi a} \times Y\right)}$$

The determination of the integral above needs a numerical treatment because the geometry correction factor, Y, becomes frequently a complex function of the crack size, a.

Subsequent stages of the fatigue life prediction method based on the crack growth analysis are shown graphically in the Figure.

# Constant Amplitude Cyclic Load - *Notation*




$S_{min}$ - minimum stress

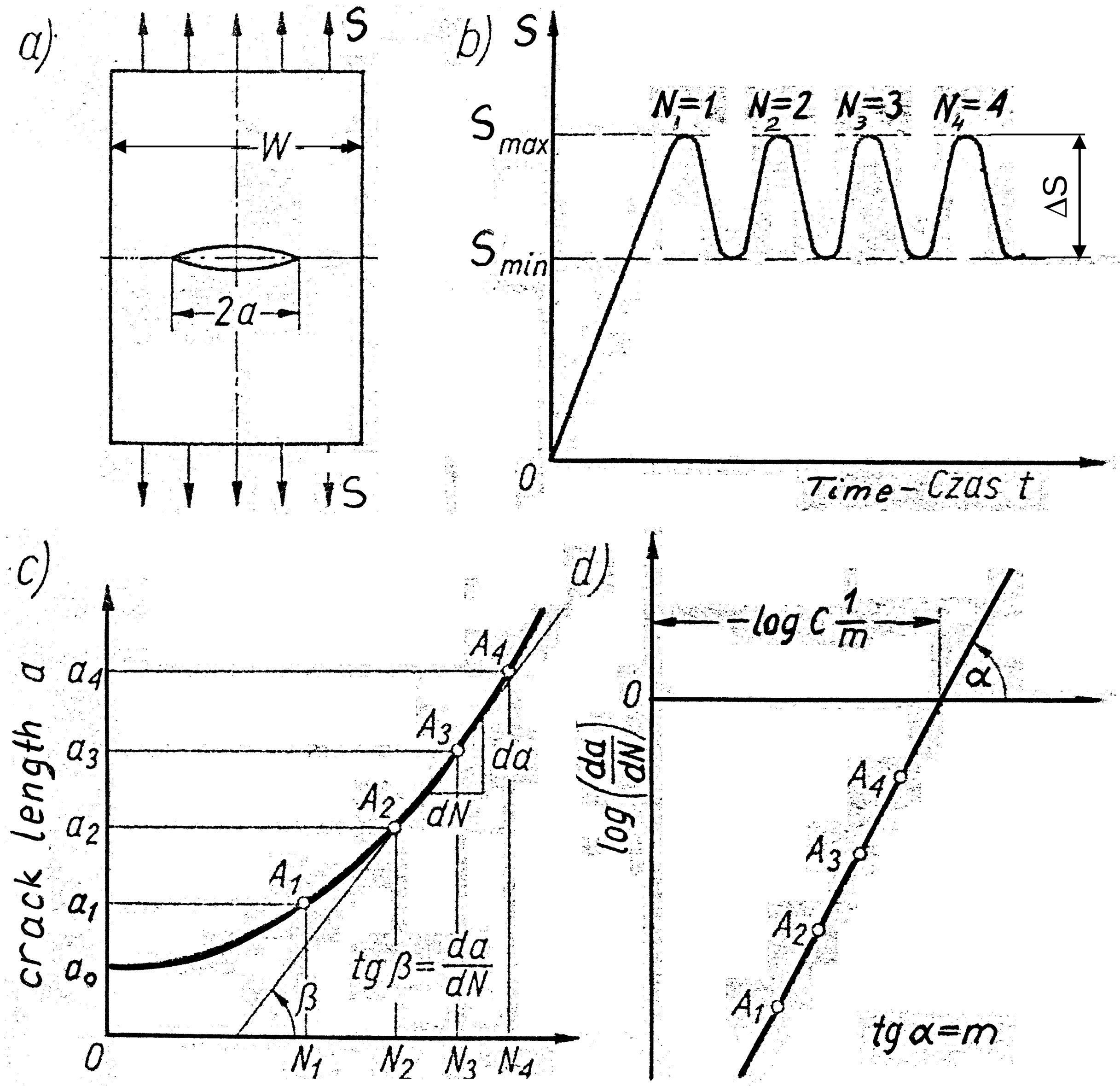
$S_{max}$ - maximum stress

$\Delta S = S_{max} - S_{min}$ - stress range

$S_a = \Delta S /2 = (S_{max} - S_{min})/2$ - mean stress

$R = S_{min} / S_{max}$ - stress ratio

# Fatigue Crack Growth Rate vs. Stress Intensity Factor

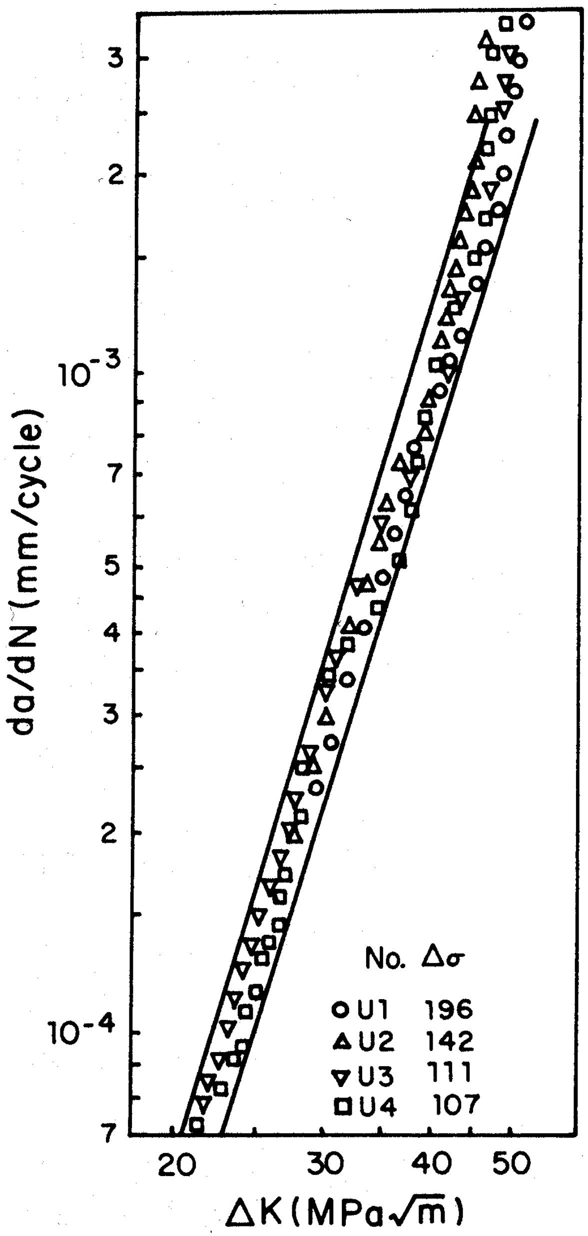



$$K = S\sqrt{\pi a} \cdot Y$$

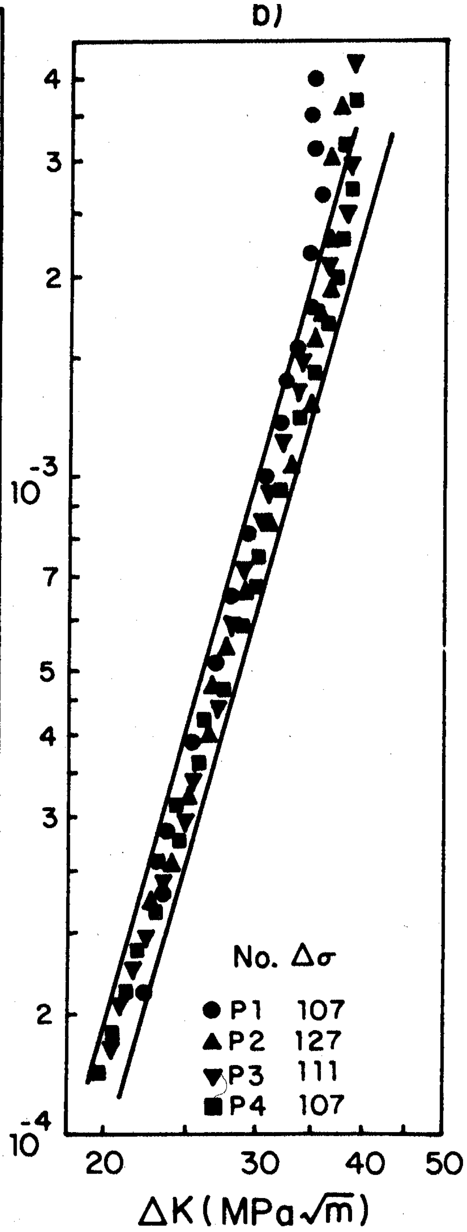
and

$$\Delta K = \Delta S\sqrt{\pi a} \cdot Y$$

## Scatter of fatigue crack growth data; Low alloy steel 18G2VA




4 mm thick plate




4 mm thick welded plate

# The Fatigue Crack Growth Expression – The Paris equation

The first mathematical relationship relating fatigue crack growth rate and the stress intensity range was proposed by Paris and Erdogan. This relationship is up to date the most popular mathematical expression used if various fatigue/fracture mechanics analyses. It was obtained by fitting power law curve into the experimental data.

$$\frac{da}{dN} = C\left(\Delta K\right)^m$$

Where: **da/dN** - fatigue crack growth rate [in/cycle or m/cycle]

**C** - Paris' equation parameter (valid for given R)

**m** - Paris' equation exponent

**ΔK** - stress intensity range

$$\Delta K = K_{\max} - K_{\min} \qquad for \quad K_{\min} \geq 0$$

$$\Delta K = K_{\max} \qquad for \quad K_{\min} < 0$$

Where:

a - crack length/depth

$S_{max}$ - maximum stress in a stress cycle

$S_{min}$ - minimum stress in a stress cycle

$K_{max}$ - maximum stress intensity factor

$K_{min}$ - minimum stress intensity factor

Y - geometry correction factor in the stress intensity factor expression

$$K_{\max} = S_{\max}\sqrt{\pi a}\cdot Y$$

$$K_{\min} = S_{\min}\sqrt{\pi a}\cdot Y$$

# Complete Fatigue Crack Growth Rate Curve, da/dN - ΔK

Soon after the Paris equation gained wide acceptance as a tool for fatigue crack growth analysis, it was found that the simple expression proposed by Paris and Erdogan had some limitations. As the Figure below illustrates the complete log-log plot of da/dN vs. ΔK is sigmoidal rather then linear and limited by the threshold stress intensity range, $\Delta K_{th}$, and the critical stress intensity factor $K_c$.

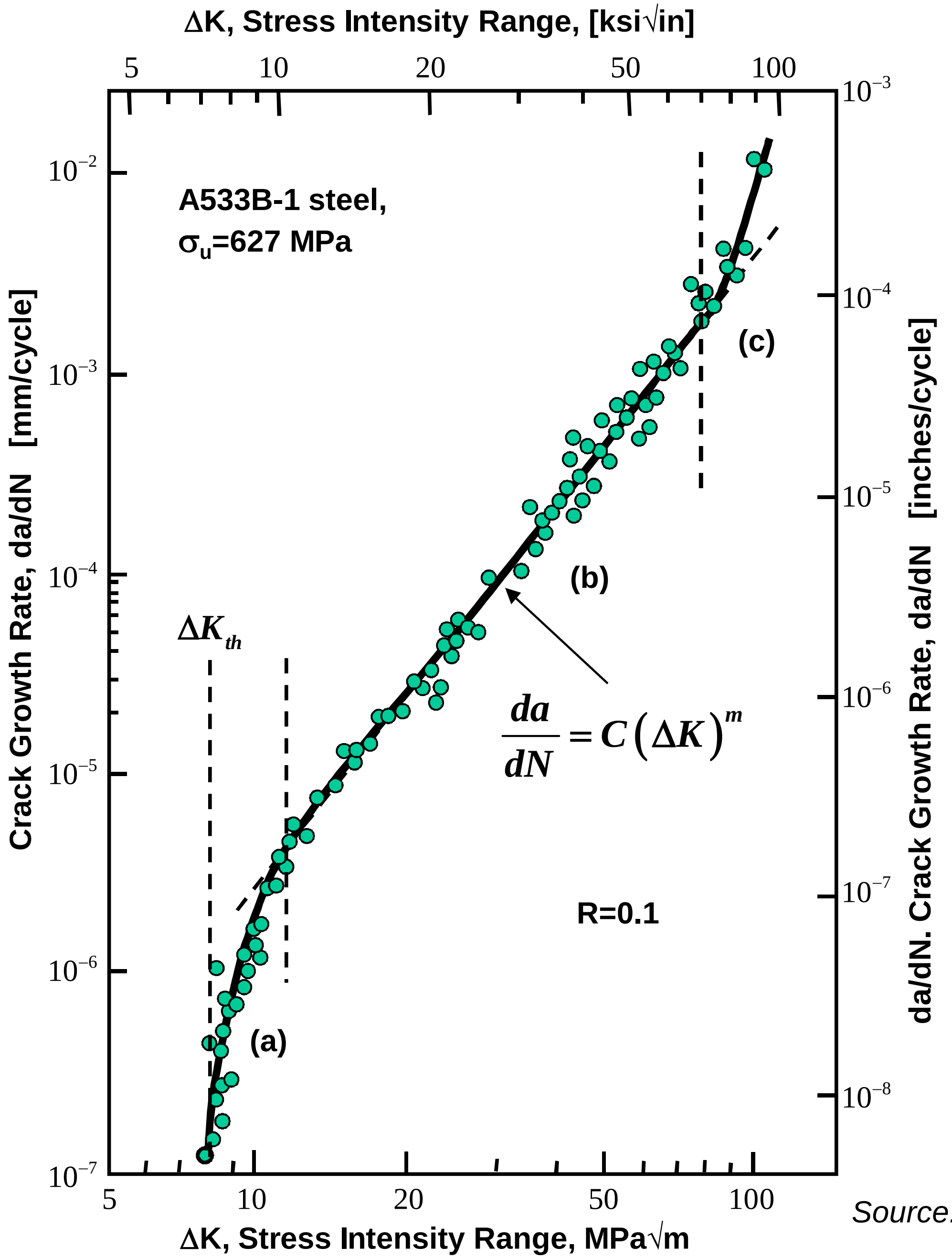
At low growth rates, the *da/dN vs. ΔK* curve becomes steep and appears to approach a vertical asymptote denoted $\Delta K_{th}$, which is called the *fatigue threshold stress intensity range* or *fatigue crack growth threshold*. This quantity is interpreted as a lower limiting value of the stress intensity factor range ΔK below which fatigue crack growth does not ordinarily occur. The fatigue crack growth threshold is analogous to the fatigue limit in the S-N approach.

At high growth rates, the *da/dN vs. ΔK* curve may again become steep. This is due to rapid unstable crack growth just prior to final fracture when $K_{max} \rightarrow K_c$. The increase of the fatigue crack rate near the critical stress intensity factor $K_c$ is due to mixture of static (monotonic -fracture) and fatigue mechanisms driving the crack growth.

Also, the fatigue crack growth rate exhibits a dependence on the stress ratio 'R'. The stress ratio R affects the fatigue crack growth rate in a manner analogous to the effects observed in the S-N and ε-N methods, i.e. for a given ΔK, increasing R-ratio increases the fatigue crack growth rate, and vice-versa.

The effect of the R -ratio (or mean stress) on Fatigue Crack Growth is most often explained using the phenomenon discovered by Elber. By measuring the compliance of specimens with fatigue cracks he noticed that the crack tip got closed during the descending part of the stress cycle in spite of the fact that the applied stress/load remained tensile (see Figure). Elber postulated that crack closure decreases the fatigue crack growth rate by reducing the effective stress intensity range.

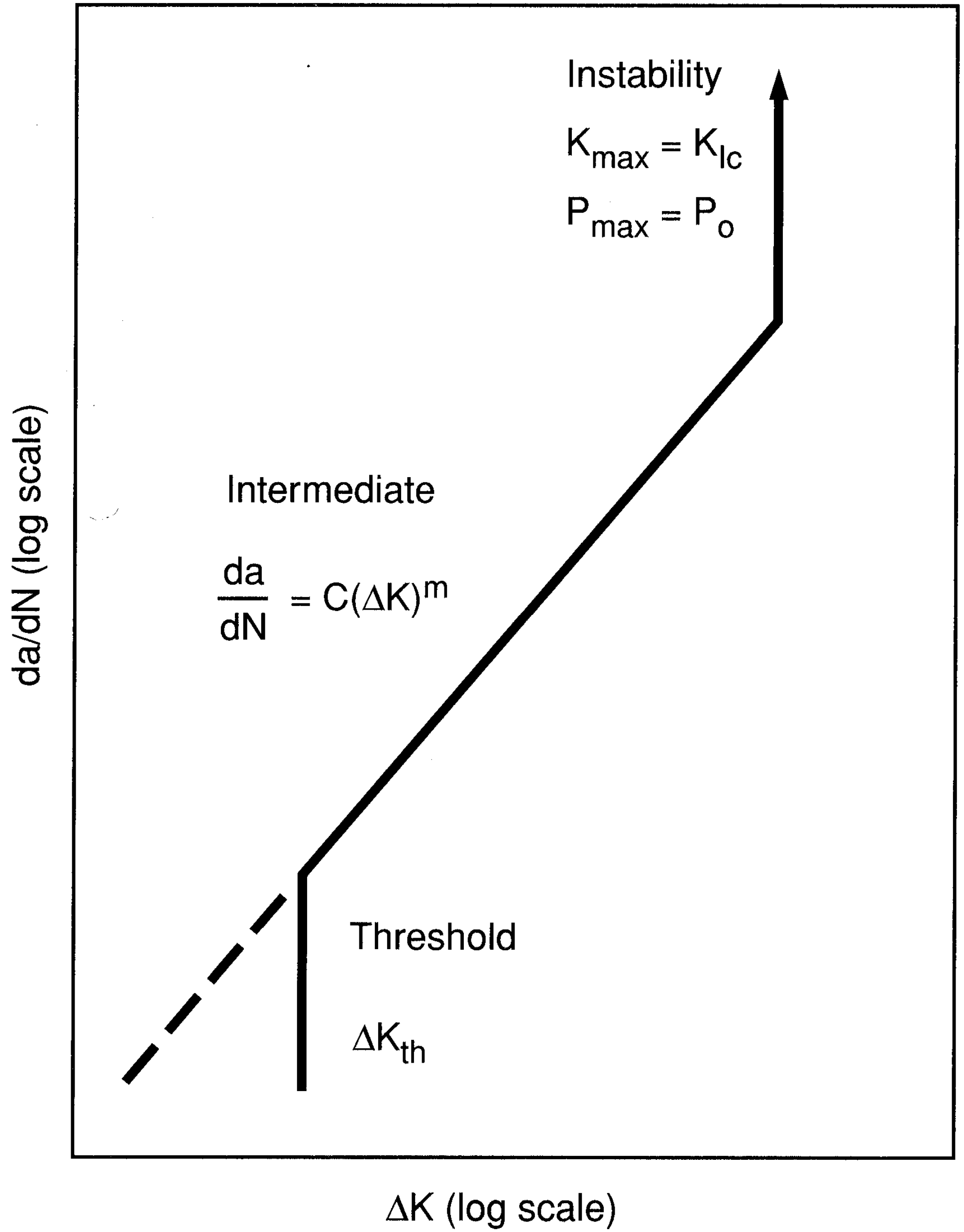
# Fatigue crack growth rates for a ductile pressure vessel steel *(the Paris equation)*




$$\frac{da}{dN} = C\left(\Delta K\right)^m$$

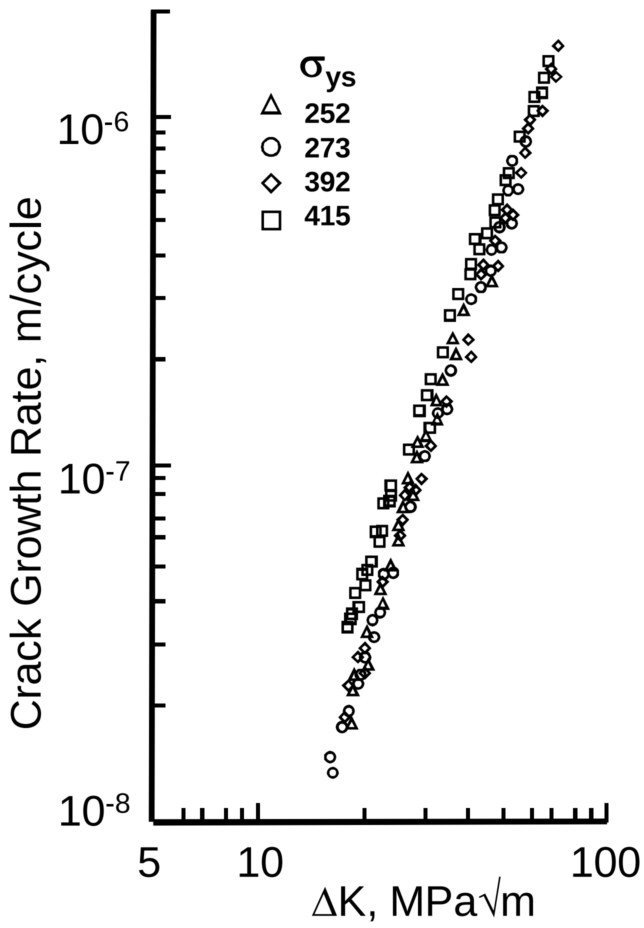
The da/dN-ΔK curve is the fatigue material curve independent of the geometry, i.e. the same curve for all geometrical crack-body configurations!

*Source: N. Dowling, ref(2)*




**For simplicity reasons the complete fatigue crack growth rate is usually approximated by three linear pieces with the two of them being vertical limiting asymptotes.**

# Paris' equation constants for steel materials at R = 0




*Ferritic-Pearlitic Steel:*

$$\frac{da}{dN} = 6.9 \times 10^{-12} \left(\Delta K\right)^{3.0}$$

*Martensitic Steel:*

$$\frac{da}{dN} = 1.4 \times 10^{-10} \left(\Delta K\right)^{2.25}$$

*Austenitic Stainless Steel:*

$$\frac{da}{dN} = 5.6 \times 10^{-12} \left(\Delta K\right)^{3.25}$$

***for: da/dN in [m/cycle] and ΔK in [MPa√m]***

*J. Barsom, "Fatigue Crack Propagation in Steels of Various Yield Strengths"*
*Journal of Engineering for Industry, Trans. ASME, Series B, Vol. 93, No. 4, 1971, 1190-1196*

# Estimation of the Fatigue Crack Propagation Life

**Basic Steps:**

1. Estimate the initial crack size and shape, $a_o$;

   - non-destructive testing - $a_o$

   - proof load - $a_o$

2. Estimate the critical crack size ac based on the fracture toughness $K_{IC}$, i.e. the crack size that the component will tolerate when the applied stress reaches its maximum $S_{max}$.

$$K_{IC} = S_{\max}\sqrt{\pi a_c}\,Y_c \Rightarrow a_c = \frac{1}{\pi}\left(\frac{K_{IC}}{S_{\max}\cdot Y_c}\right)^2$$

3. Using the same expression for the stress intensity factor calculate the stress intensity range $\Delta$K.

$$\Delta K = \Delta S\sqrt{\pi a}\,Y \qquad for \quad R \geq 0$$

$$\Delta K = S_{\max}\sqrt{\pi a}\,Y \quad for \quad R < 0 \quad (if \;\; \sigma_r \leq 0!!)$$

4. Substitute ΔK into fatigue crack growth equation (Paris or Forman)

$$\frac{da}{dN} = C(\Delta K)^m = C(\Delta S\sqrt{\pi a})^m Y^m$$

5. Integrate the equation above from $a = a_o$ to $a = a_c$ and determine the number of cycles, N, necessary to grow the crack from the initial crack size of $a_o$ to the critical size of $a_c$. This is the estimated fatigue crack propagation life of given component!

$$dN = \frac{da}{C(\Delta K)^m}$$

$$N = \int_{a_o}^{a_c} \frac{da}{C(\Delta K)^m} = \int_{a_o}^{a_c} \frac{da}{C(\Delta S\sqrt{\pi a}\,Y)}$$

Note! In most practical cases the integration requires numerical solution due to the complexity of the geometric factor Y.

# Integrated Paris' Equation for a Constant Geometric Factor, Y = const.

$$\frac{da}{dN} = C\left(\Delta K\right)^m = C(\Delta S\sqrt{\pi a}Y\ )^m$$

$$for \quad m \neq 2$$

$$N = \frac{2}{(m-2)C(\Delta SY)^m \pi^{m/2}}\left[\frac{1}{a_o^{(m-2)/2}} - \frac{1}{a_c^{(m-2)/2}}\right];$$

$$for \ m = 2$$

$$N = \frac{1}{C\Delta S^2 \pi Y^2}\ln\frac{a_c}{a_o};$$

# Numerical Integration of the Paris Equation

**If the Y factor is not constant a numerical technique has to be applied. The most often used is the cycle by cycle technique based on the calculation of crack increments $\Delta a_i$ corresponding to each load cycle. In this case, the infinitesimal increments da and dN are replaced by finite differences $\Delta a$ and $\Delta N= 1$.**

$$\frac{\Delta a_i}{\Delta N_i} = C(\Delta K_i)^m = C(\Delta S_i \sqrt{\pi a_{i-1}}\ Y_{i-1})^m; \;\; a_i = a_o + \sum_{i=1}^{N} \Delta a_i; \;\; \Delta a_i = C(\Delta S_i \sqrt{\pi a_{i-1}} Y_i)^m \Delta N_i$$

$$N_0 = 0 \qquad \Delta a_0 = 0 \qquad a_0 = a_0;$$

$$N_1 = 1; \;\; \Delta N_1 = 1; \;\; \Delta a_1 = C\left(\Delta S_1 \sqrt{\pi a_0}\ Y_0\right)^m; \;\; a_1 = a_0 + \Delta a_1;$$

$$N_2 = 1; \;\; \Delta N_2 = 1; \;\; \Delta a_2 = C\left(\Delta S_2 \sqrt{\pi a_1}\ Y_1\right)^m; \;\; a_2 = a_1 + \Delta a_2;$$

$$N_3 = 1; \;\; \Delta N_3 = 1; \;\; \Delta a_3 = C\left(\Delta S_3 \sqrt{\pi a_2}\ Y_2\right)^m; \;\; a_3 = a_2 + \Delta a_3;$$

$$N_4 = 1; \;\; \Delta N_4 = 1; \;\; \Delta a_4 = C\left(\Delta S_4 \sqrt{\pi a_3}\ Y_3\right)^m; \;\; a_4 = a_3 + \Delta a_4;$$

........................................................................................

$$N_i = 1; \;\; \Delta N_i = 1; \;\; \Delta a_i = C\left(\Delta S_i \sqrt{\pi a_{i-1}}\ Y_{i-1}\right)^m; \;\; a_i = a_{i-1} + \Delta a_i;$$

$$until \quad a_i \le a_c$$

**Calculations have to be carried out for each cycle !!**

## Subsequent stages of fatigue life prediction method based on the crack growth analysis

- Analysis of external forces acting on the structure and the component in question (a),
- Analysis of internal loads in chosen cross section of a component ( b),
- Selection of individual welded joints in the structure (c),
- Identification of appropriate nominal or reference stress history (d),
- Extraction of stress cycles (rainflow counting) or reversals from the stress history (Fig.e),
- Determination of the stress intensity factor (i.e. the factor Y) for postulated or existing crack,

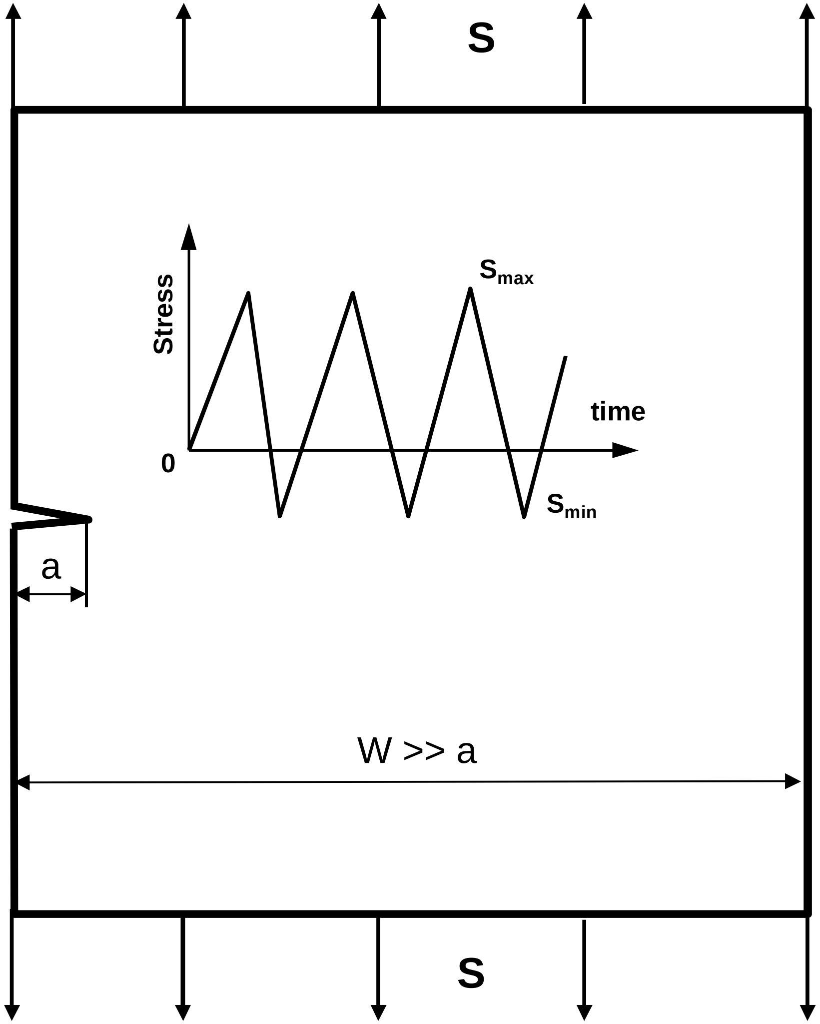
  - indirect method (Fig.f):

  - ➢ analyze un-cracked weldment and determine the stress field, $\sigma(x,y)$, in the prospective crack plane; normalize the calculated stress distribution with respect to the nominal or any other reference stress, i.e. $\sigma(x,y)/\sigma_n$,
  - ➢ choose appropriate weight function, calculate stress intensity factor
  - ➢ determine the stress or displacement field near the crack, or the strain energy release rate,
  - ➢ calculate stress intensity factor using.
- Determination of crack increments for each stress cycle (Fig. h),
- Determination of the number of cycles, N, necessary to grow the crack from its initial size, $a_0$, up to the final size, $a_f$.

A summary of necessary input data and procedures used in the, da/dN - $\Delta K$, approach to fatigue life estimation is also presented in the Figure.

**Example**: A very wide SAE 1020 cold-rolled thin plate is subjected to constant amplitude uni-axial cyclic loads that produce nominal stresses varying from $S_{max}$=200MPa (29ksi) to $S_{min}$=-50 MPa (-7.3ksi). The monotonic properties for this steel are $\sigma_Y$=630 MPa (91 ksi), $\sigma_{uts}$=670 MPa (97 ksi), E=207000 MPa (30000 ksi), $K_c$ =104 MPa$\sqrt{}$m (95 ksi$\sqrt{}$in). What fatigue life would be attained if an initial through-thickness edge crack existed and was 1 mm (0.04 in) in depth?

The fatigue crack growth data are: $\Delta K_{th(r=0)}$=6 MPa$\sqrt{}$m, and Paris' equation parameters C=6.9×$10^{-12}$ and m=3.



**A. What is the stress intensity factor expression?**

Semi-infinite plate with an edge crack.

$$K_{\max} = S_{\max}\sqrt{\pi a}\cdot Y = S_{\max}\sqrt{\pi \times a}\times 1.12$$

**B. Is Linear Elastic Fracture Mechanics (LEFM) applicable?**

$$\textit{Nominal stress level}:$$

$$S_{\max} < 0.8\sigma_Y = 0.8\times 630 = 504\,MPa \quad -YES!$$

$$\textit{Plastic zone size}:$$

$$K_{\max} = S_{\max}\sqrt{\pi a}\cdot Y = 200\sqrt{\pi\times 0.001}\times 1.12 = 12.6\,MPa\sqrt{m}$$

$$r_y = \frac{1}{2\pi}\left(\frac{K_{\max}}{\sigma_Y}\right)^2 = \frac{1}{\pi}\left(\frac{12.6}{630}\right)^2 = 0.0000635\,m = 0.0635\,mm$$

$$\frac{r_y}{a} = \frac{0.0635}{1} < \frac{1}{8} = 0.125 \qquad -YES!$$

**C. The effective stress range**

$$\Delta S = S_{\max} - S_{\min} \quad for \quad S_{\min} > 0$$

$$\Delta S = S_{\max} \quad for \quad S_{\min} < 0$$

$$S_{\max} = 200 MPa \;\; and \;\; S_{\min} = -50 MPa$$

$$thus$$

$$\Delta S = S_{\max} = 200 MPa$$

**D. Is the Paris equation applicable?**

$$Paris\ equation\ is\ valid\ \ for\ \Delta K > \Delta K_{th}!$$

$$Smallest\ \Delta K = \Delta K_0\ occurs\ \ for\ a = a_0 = 0.001 m.$$

$$\Delta K_0 = \Delta S \sqrt{\pi a_0} Y = 200 \sqrt{\pi \times 0.001} \times 1.12 = 12.6 MPa\sqrt{m}$$

$$\Delta K_0 = 12.6 > \Delta K_{th} = 6 MPa\sqrt{m} \quad - YES, Paris\ equation\ is\ applicable!$$

**E. What is the critical/final crack size?**

$$K_c = K_{final} = S_{\max}\sqrt{\pi a_c}Y$$

$$a_c = \frac{1}{\pi}\left(\frac{K_c}{S_{\max}\times Y}\right)^2 = \frac{1}{\pi}\left(\frac{104}{200\times 1.12}\right)^2 = 0.068\,m = 68\,mm$$

**E. Integration of the Paris equation**

Analytical integration is possible because Y= const.

$$\frac{da}{dN} = C\left(\Delta K\right)^m = C(\Delta S\sqrt{\pi a}Y\ )^m$$

$$N = \int_{a_0}^{a_c}\frac{da}{C\left(\Delta K\right)^m} = \int_{a_0}^{a_c}\frac{da}{C(\Delta S\sqrt{\pi a}Y\ )^m} = \frac{1}{C\cdot\Delta S^m\cdot\pi^{m/2}}\int_{a_0}^{a_c}\frac{da}{a^{m/2}\cdot Y^m}$$

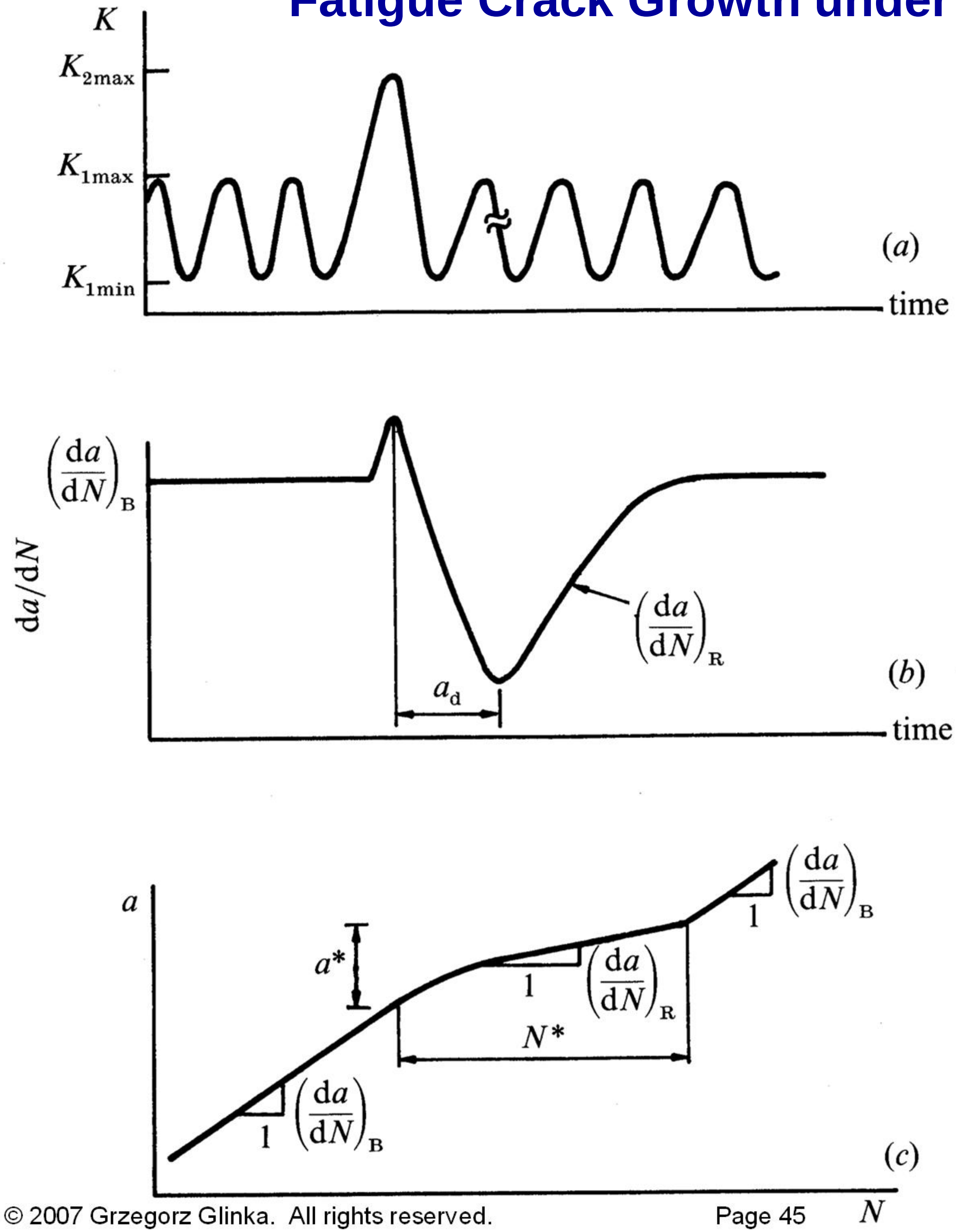
$$for\ \ m\neq 2\ and\ Y = const$$

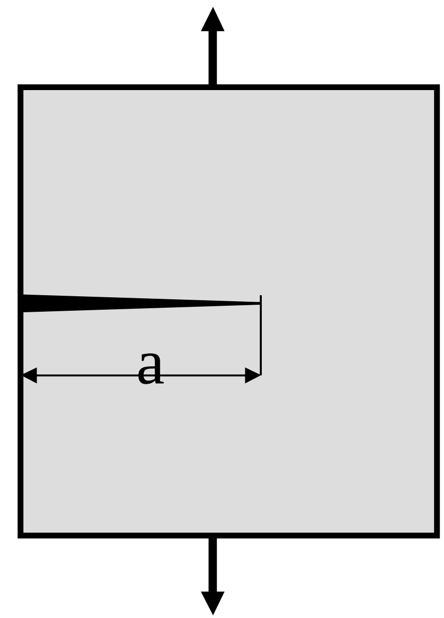
$$N = \frac{2}{(m-2)\cdot C\cdot(\Delta S\cdot Y)^m\cdot\pi^{m/2}}\left[\frac{1}{a_0^{(m-2)/2}} - \frac{1}{a_c^{(m-2)/2}}\right];$$

$$N = \frac{2}{(3-2)\cdot 6.9\times 10-12\cdot(200\cdot 1.12)^3\cdot\pi^{3/2}}\left[\frac{1}{0.001^{(3-2)/2}} - \frac{1}{0.068^{(3-2)/2}}\right]$$

$$= 4631\left[\frac{1}{a_0^{(m-2)/2}} - \frac{1}{a_c^{(m-2)/2}}\right] = 4631\left[\frac{1}{0.0316} - \frac{1}{0.2608}\right] = 4631\left[31.645 - 3.834\right] = 128792\ cycles$$

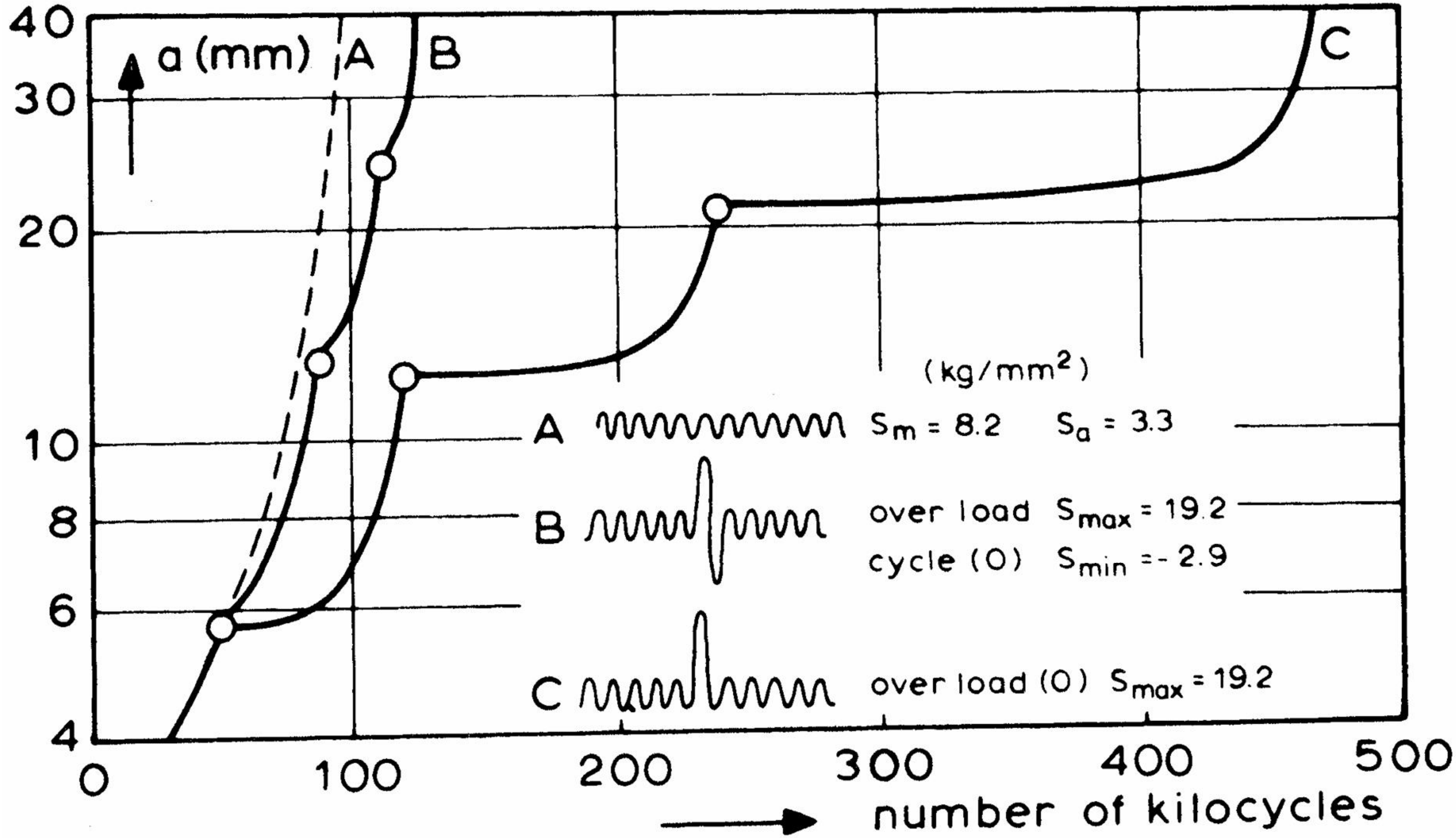
# Fatigue Crack Growth under Variable Amplitude Loading: the retardation effect



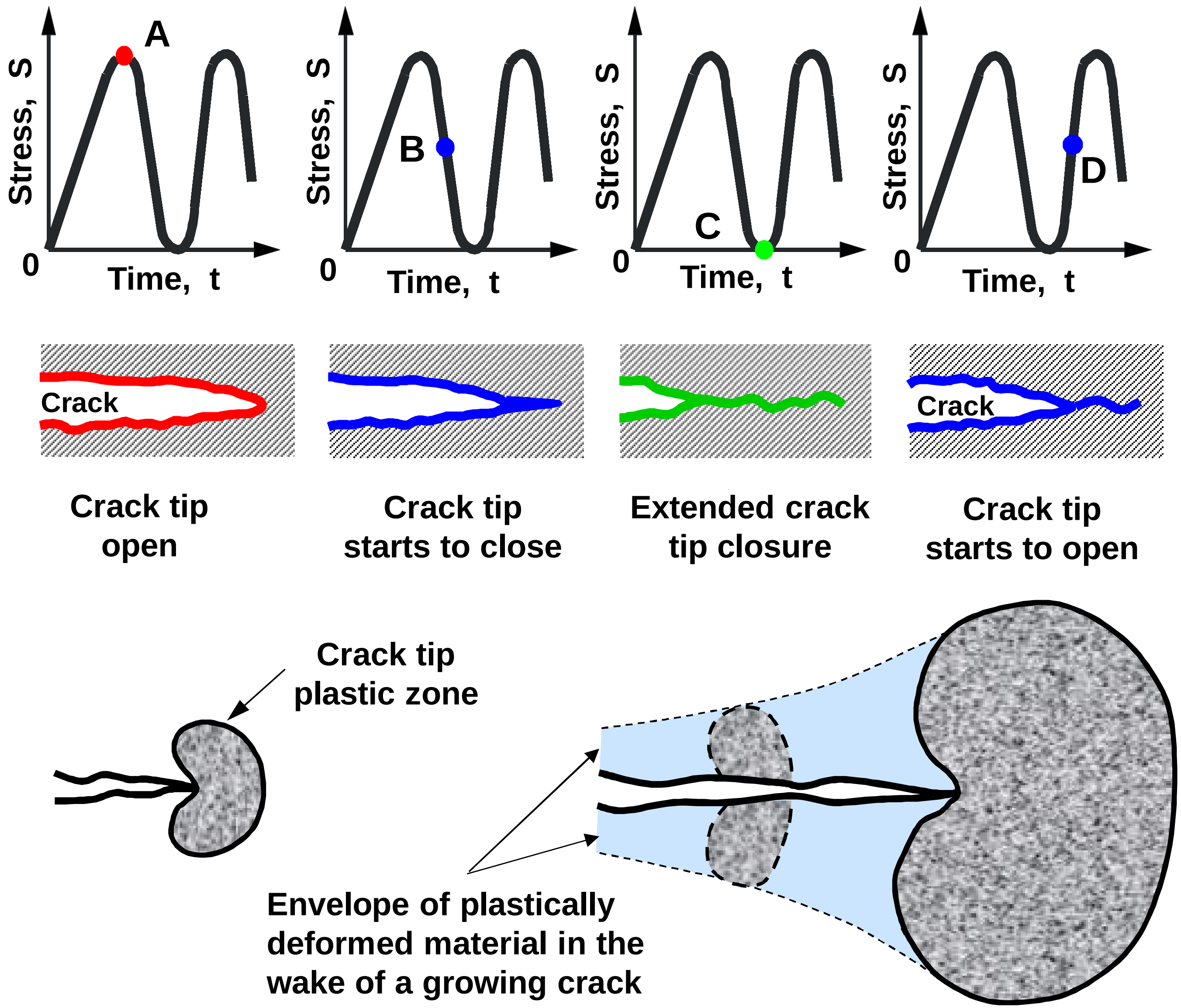

$$\frac{da}{dN} = C\left(\Delta K\right)^m$$

$$\frac{da}{dN} = f\left(\Delta K, K_{max}\right)$$

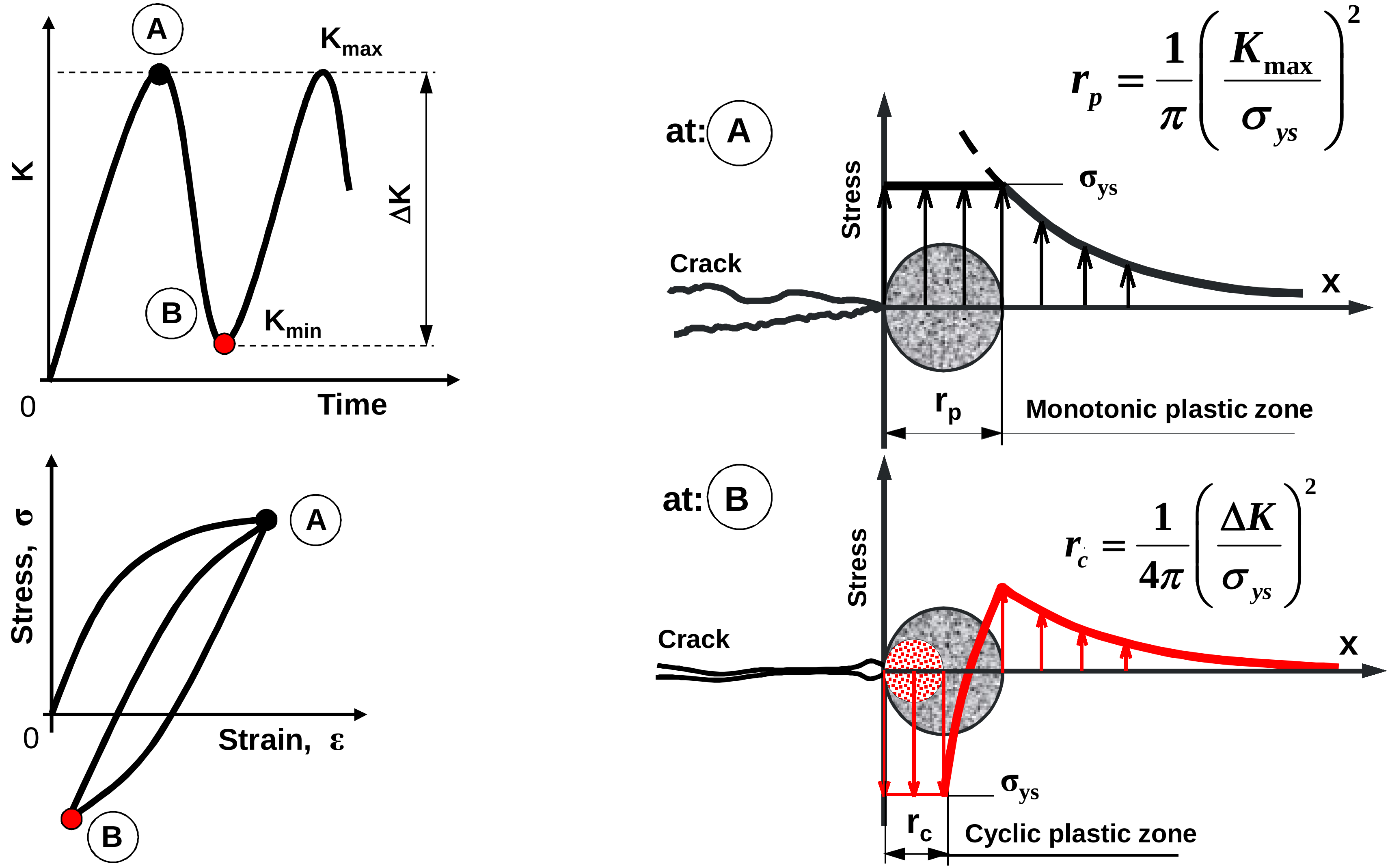



A schematic illustration of transient crack growth during constant amplitude fatigue (*A*) and during variable amplitude loading involving single tensile overloads (*C*) or tensile-compressive overload sequences (*B*). The open circles represent the crack length locations at which each variable amplitude sequence is applied.

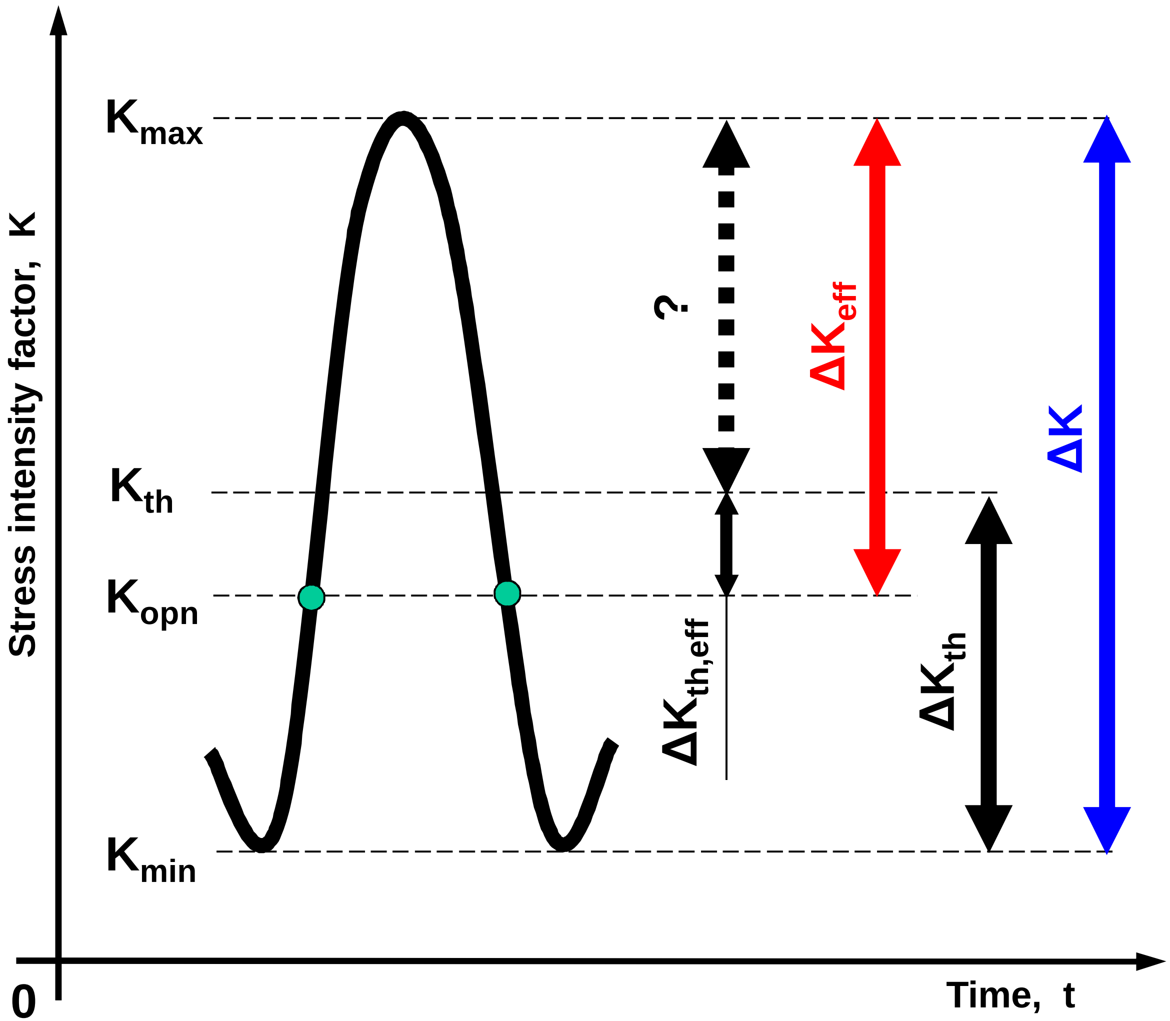
# Evolution of the crack tip plastic zone ahead of a fatigue crack & crack tip closure



# The stress-strain evolution and the monotonic and plastic zone ahead of a fatigue crack tip



# The effect of the crack tip closure




$$R = \frac{K_{\min}}{K_{\max}}$$

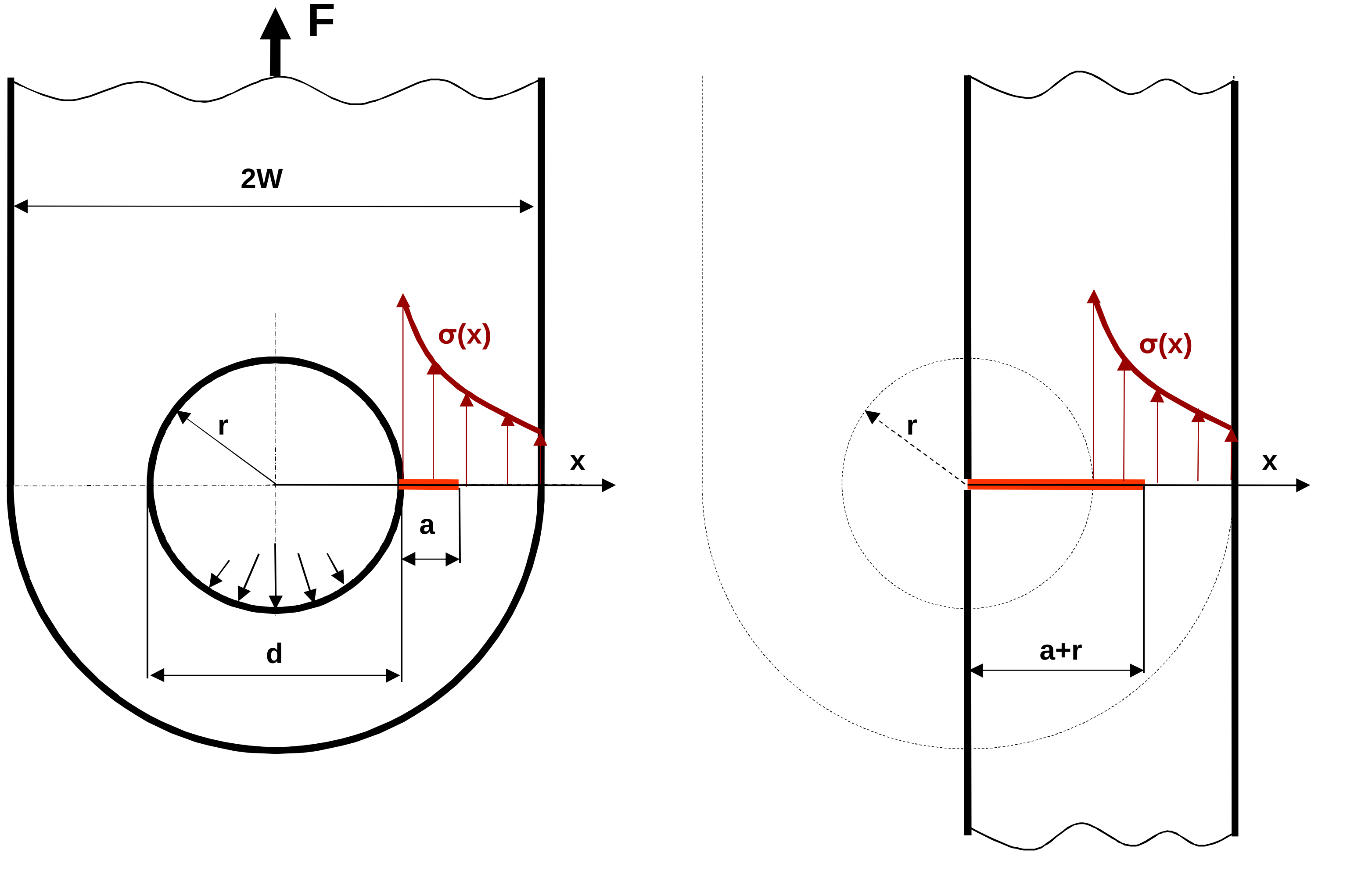
$$\gamma = \frac{K_{opn}}{K_{\max}}$$

$$U = \frac{\Delta K_{eff}}{\Delta K}$$

*or*

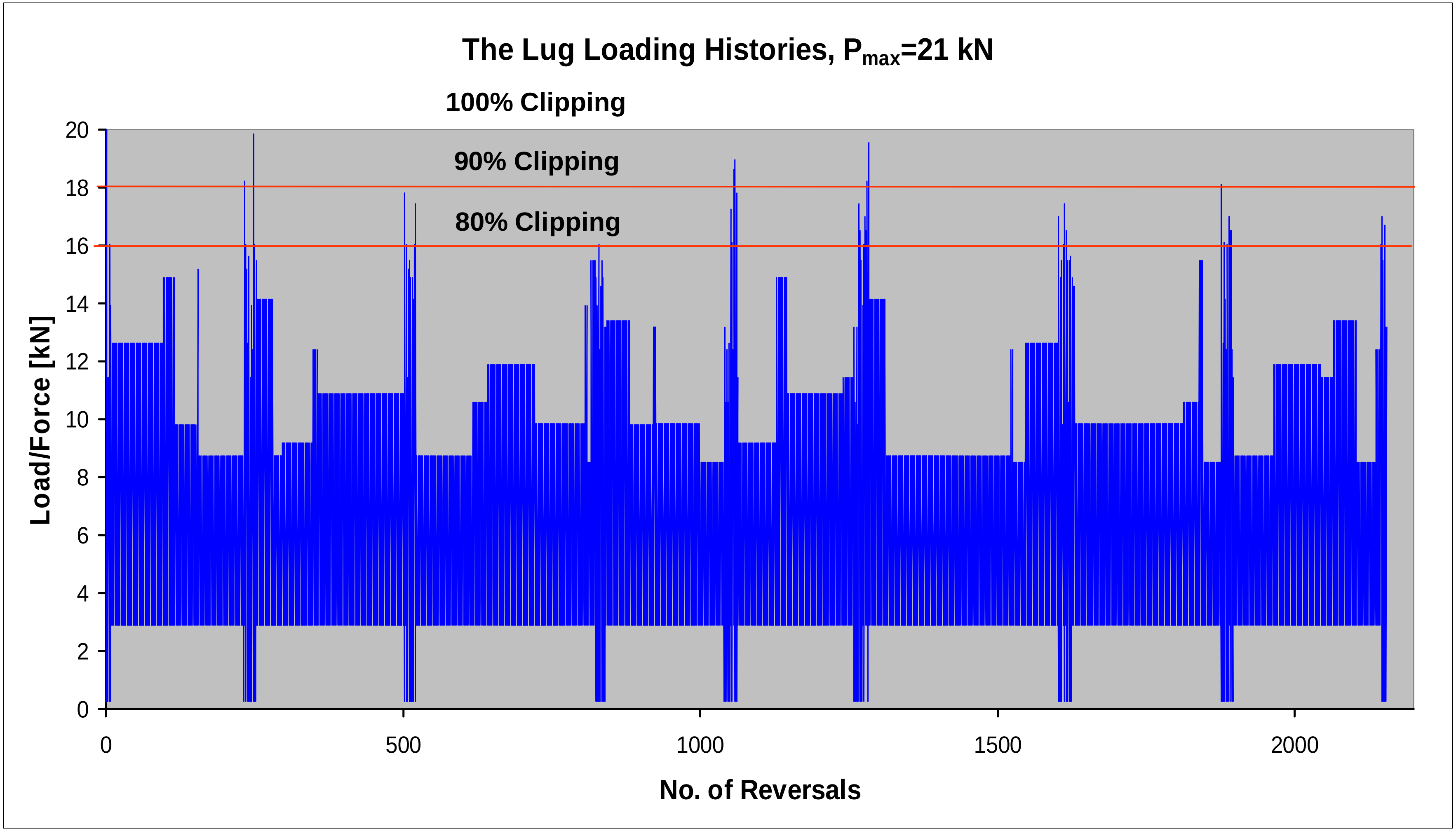
$$U = \frac{1-\gamma}{1-R}$$

# Fatigue Growth of Corner Cracks in a Lug Subjected to a VA Loading History

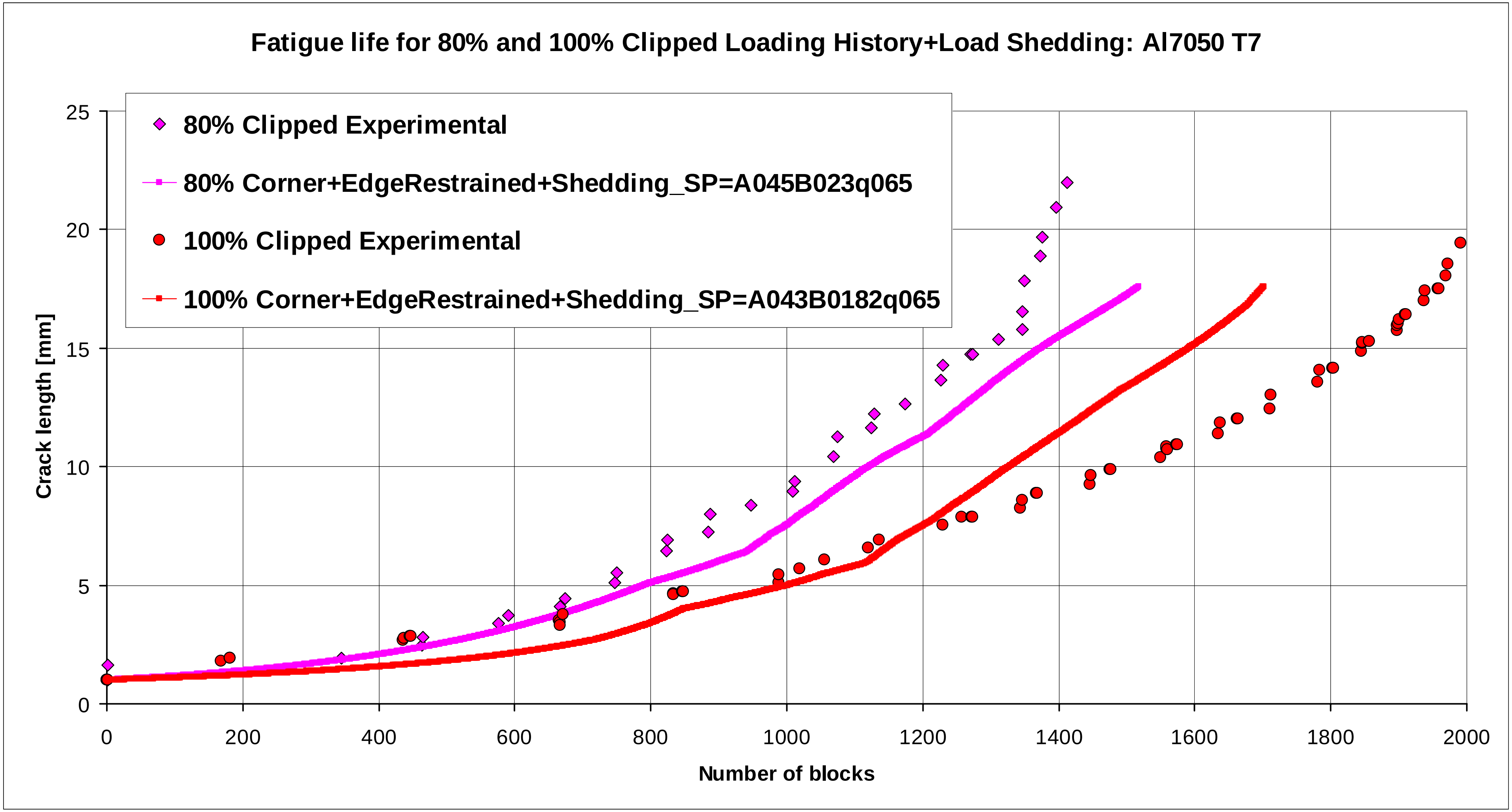



Experimental data from: **Jong-Ho Kim, Soon-Bok Lee, Seong-Gu Hong,** ***Int. Journal of Fatigue, vol. 40, 2003***

# The VA Load/Stress History

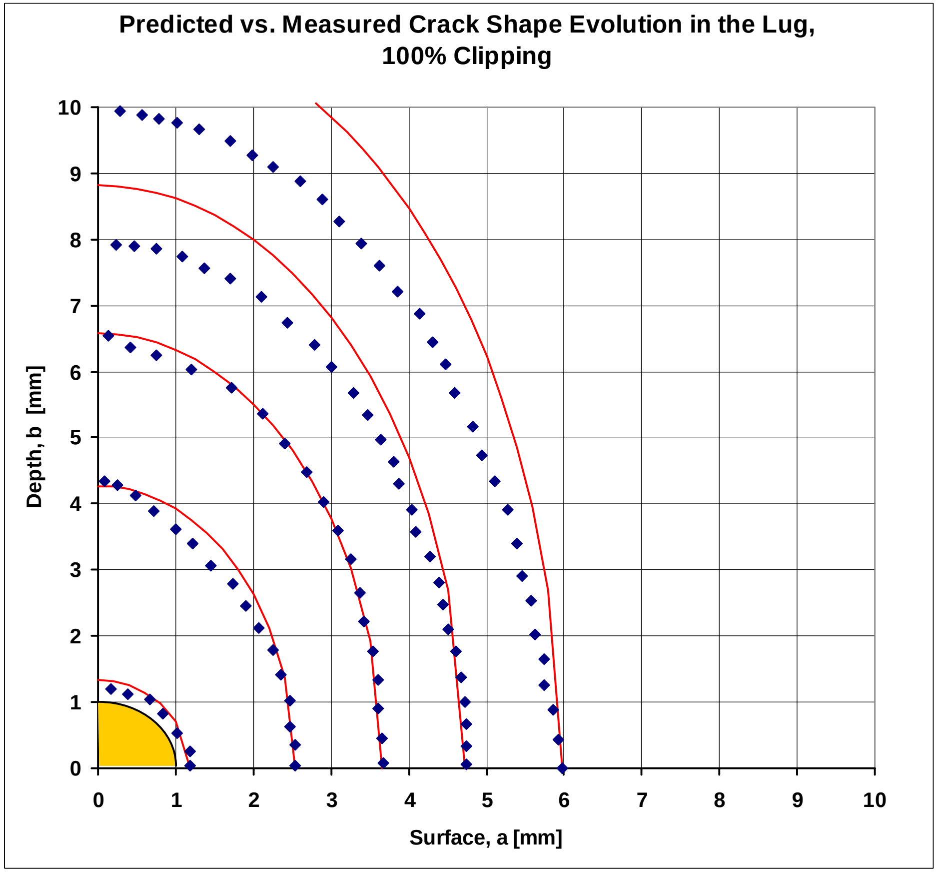


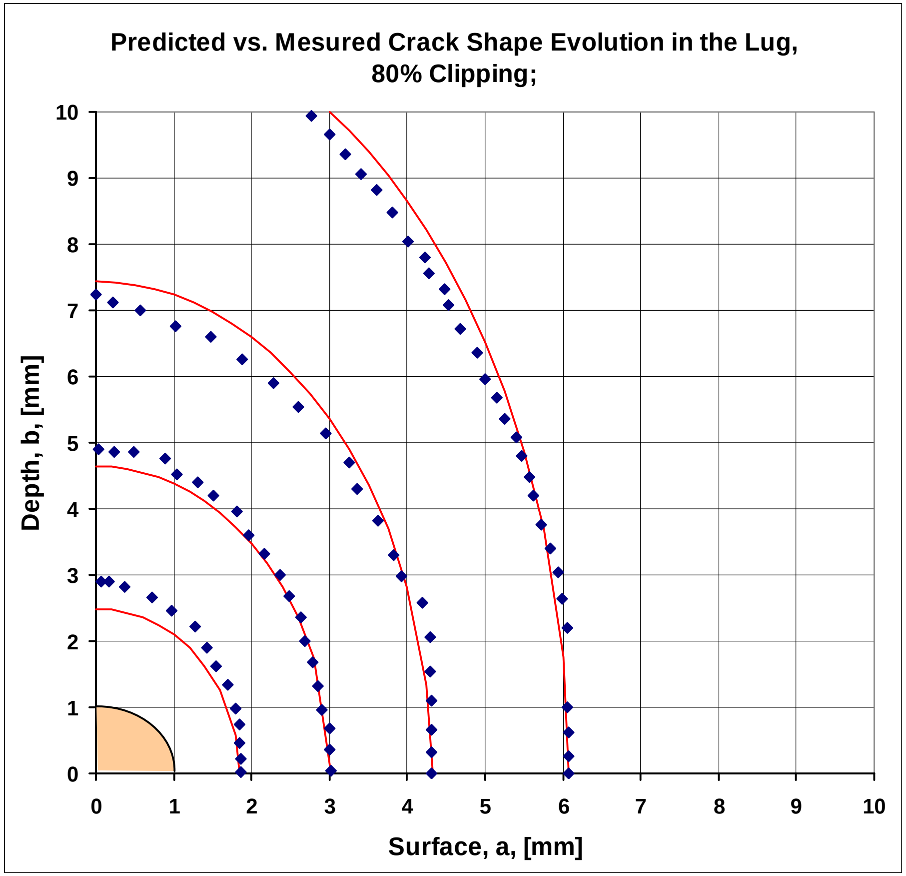
# Calculations vs. Experiment




Experimental data from: **Jong-Ho Kim, Soon-Bok Lee, Seong-Gu Hong,** ***Int. Journal of Fatigue, vol. 40, 2003***

# Crack Shape Evolution; *quarter circular initial crack*



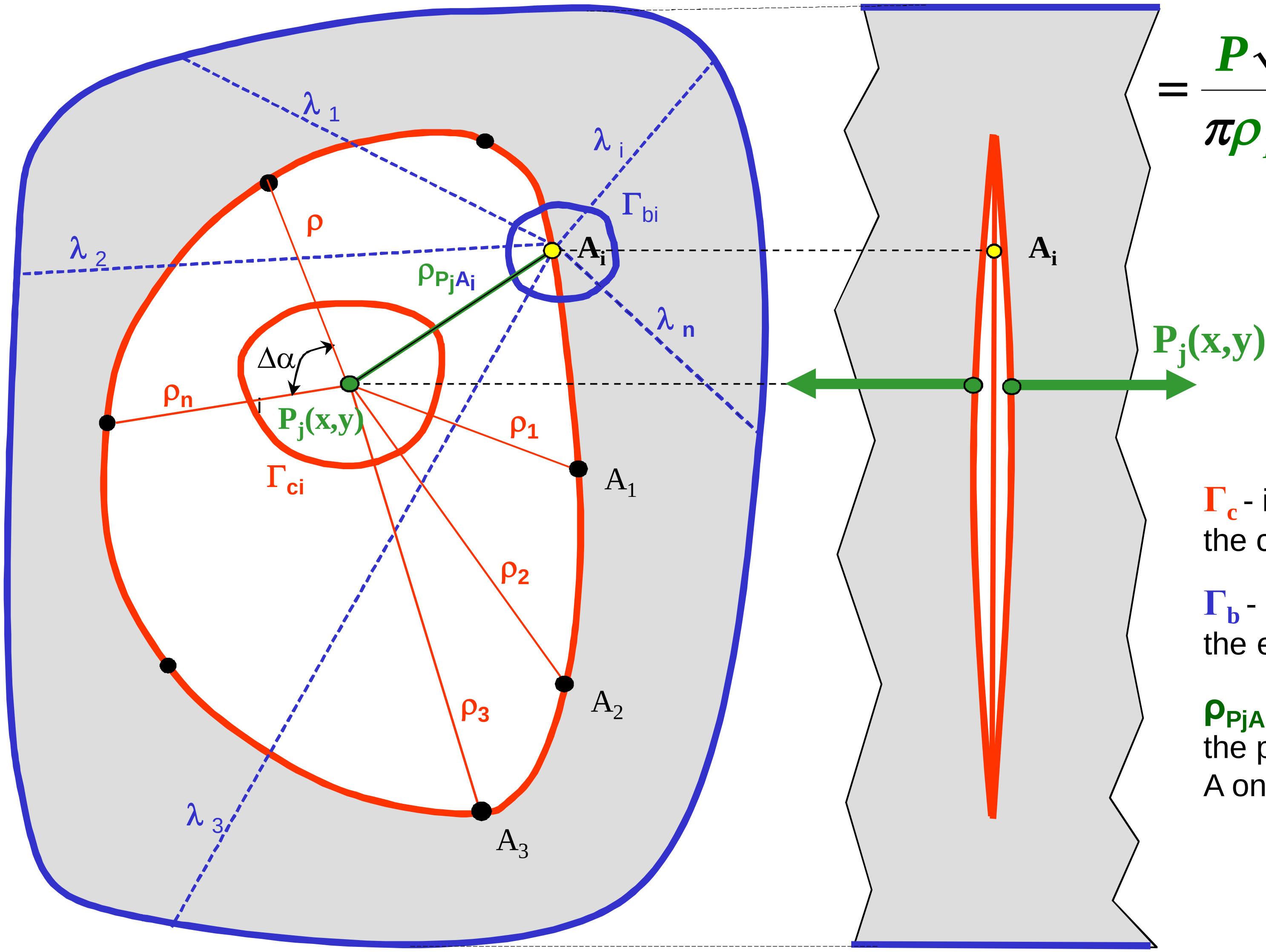




Experimental data from: **Jong-Ho Kim, Soon-Bok Lee, Seong-Gu Hong,** ***Int. Journal of Fatigue, vol. 40, 2003***

# Weight Function for Arbitrary Planar Cracks

$$K_{A_i} = m_{A_i}(x, y; P)$$

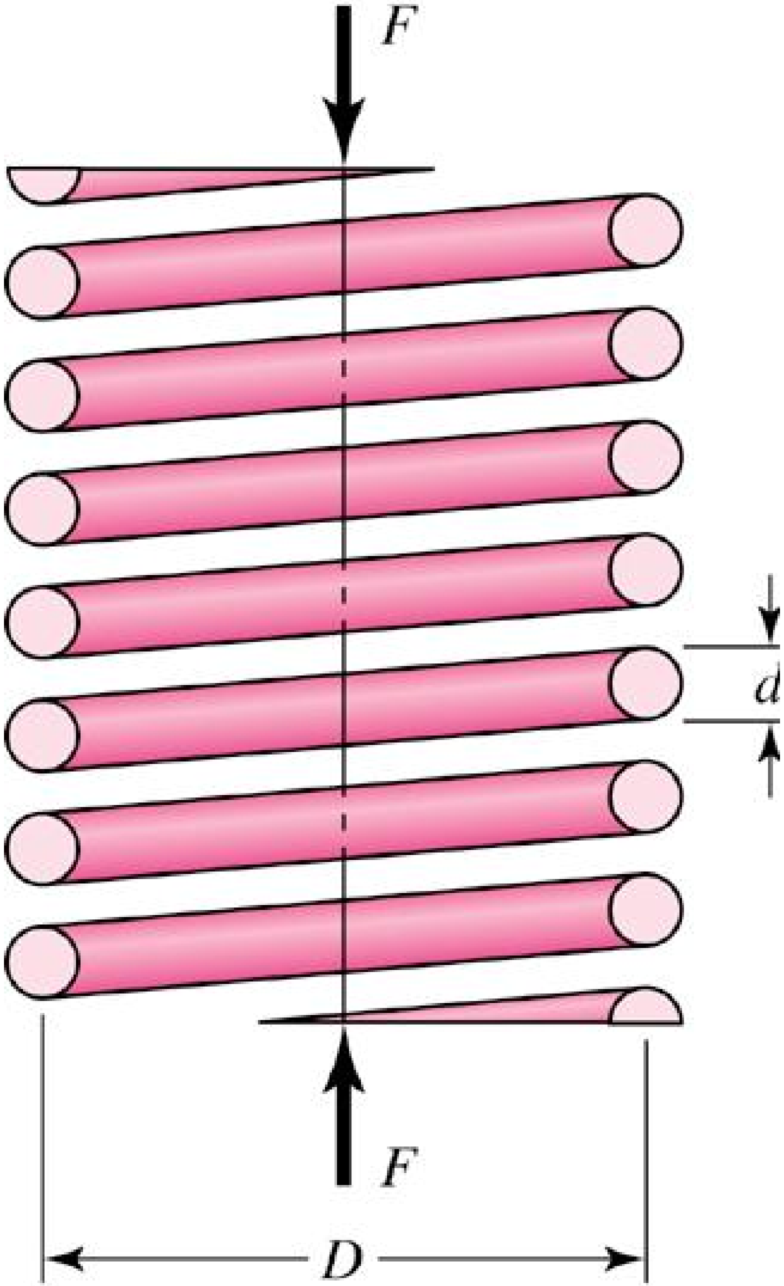
$$= \frac{P\sqrt{2}}{\pi \rho^2_{P_j A_i}} \times \frac{\sqrt{\Gamma_c + \Gamma_b}}{\Gamma_c}$$

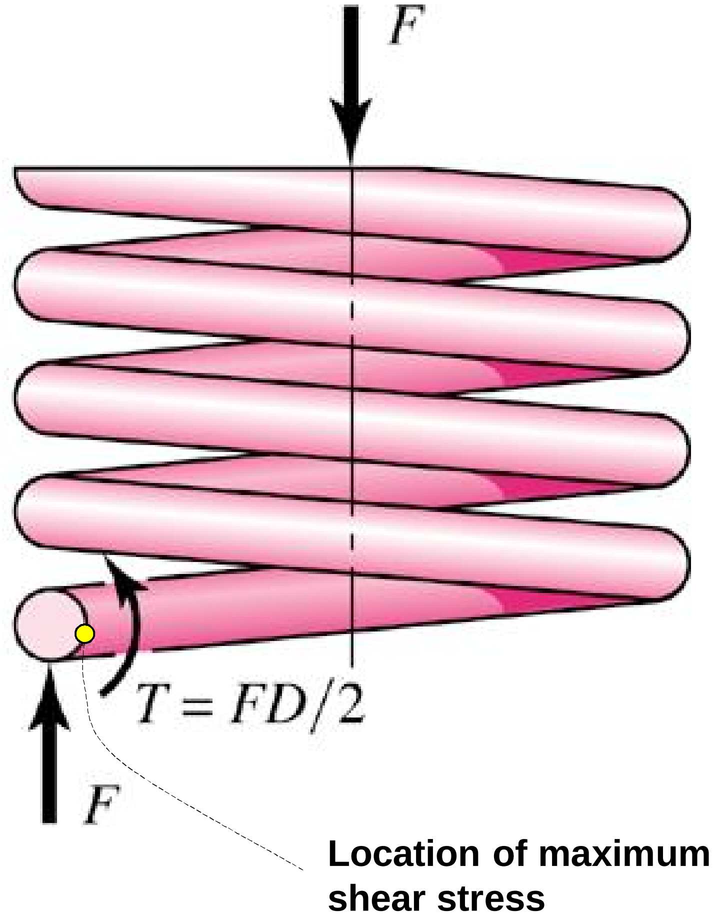



$\Gamma_c$ - inverted contour of the crack front;

$\Gamma_b$ - inverted contour of the external boundary;

$\rho_{PjAi}$ - distance between the point load P and point A on the crack front

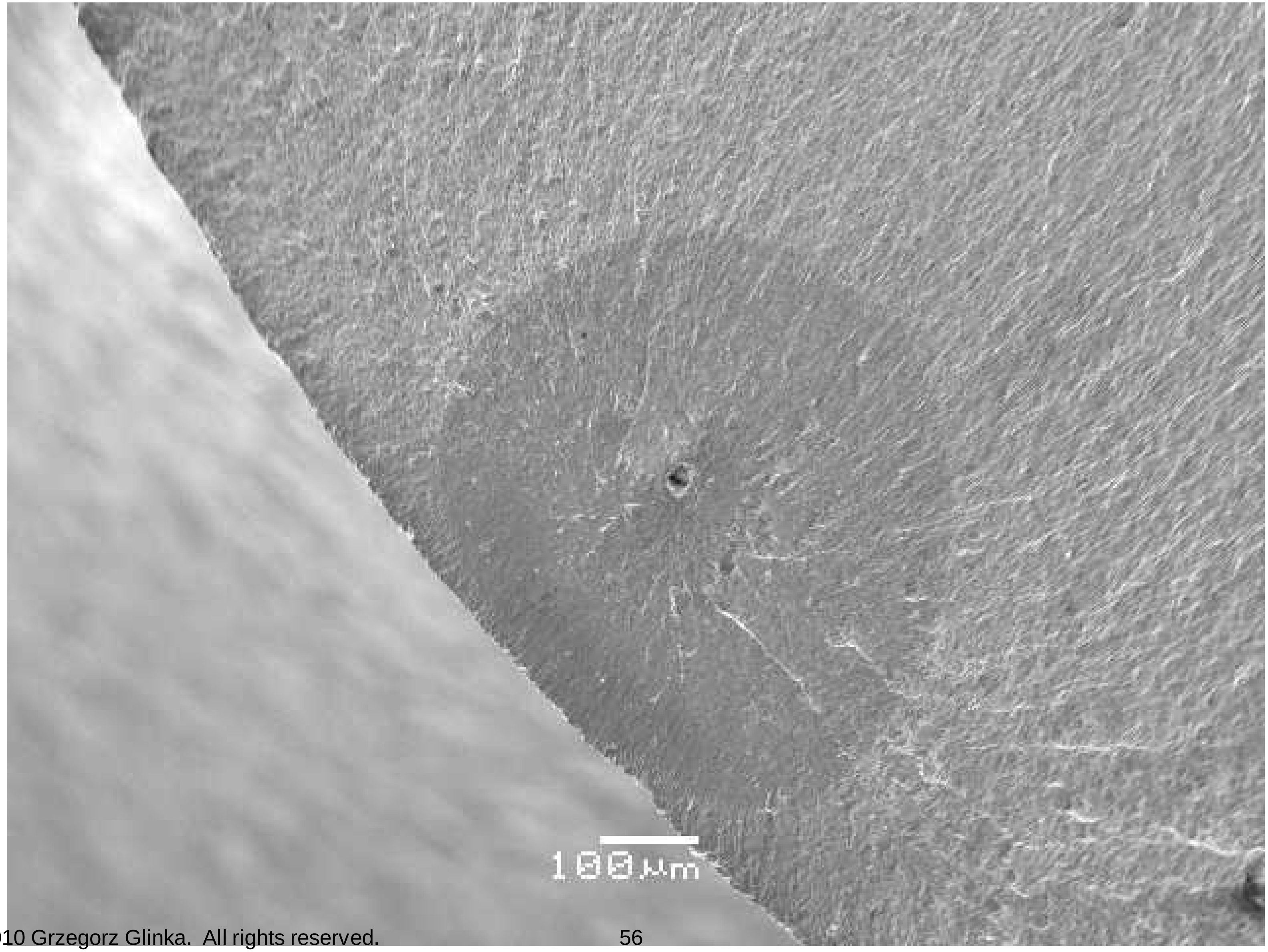



D=18.5 mm, d=4 mm

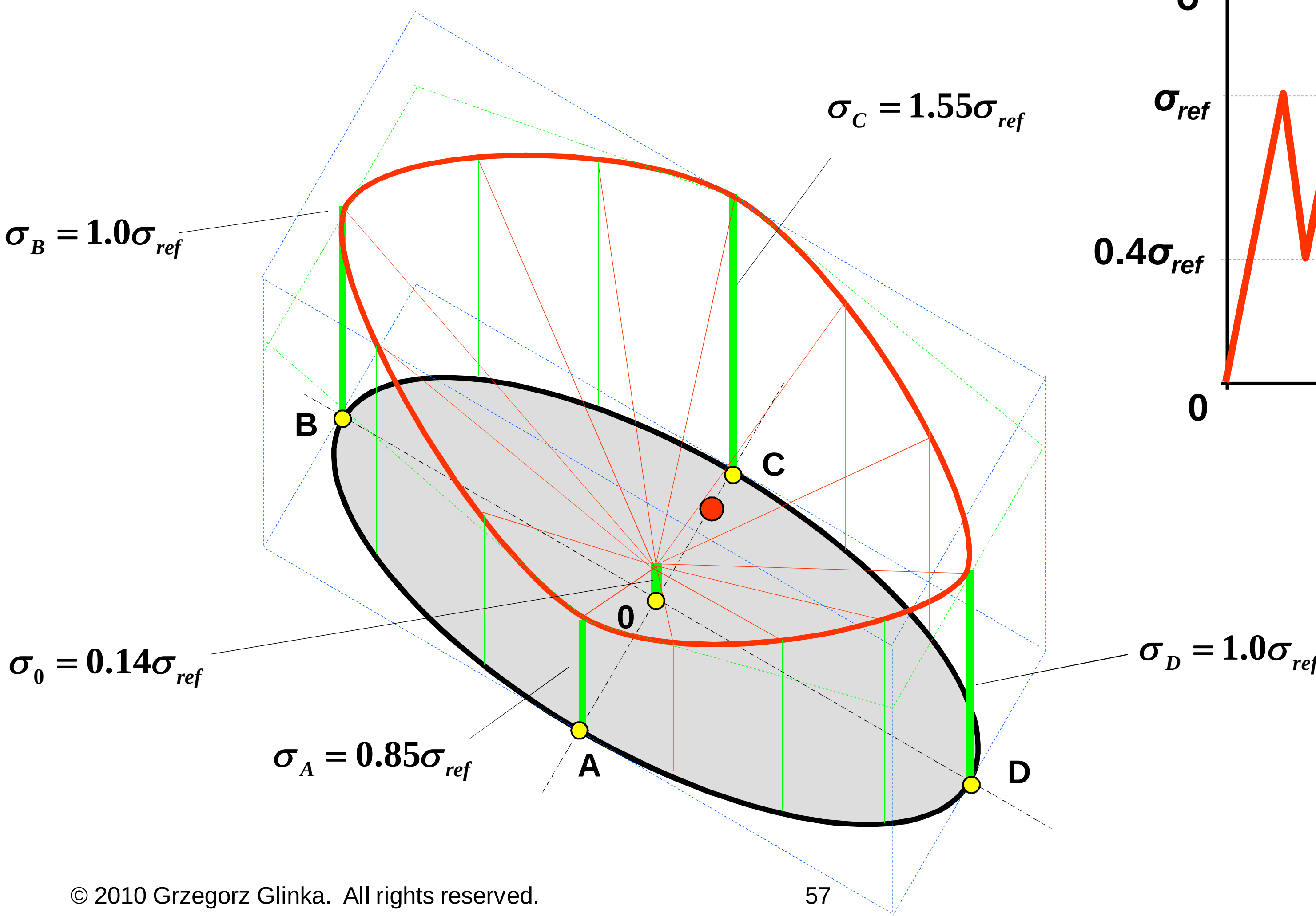
Fig. 10. Geometry and dimensions of the spring

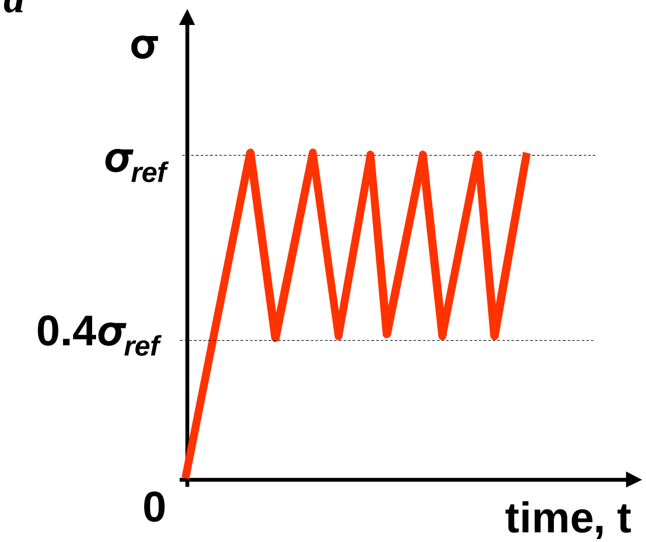
# Relative dimensions of the inclusion (d=20-30 μm) and the final crack size ($2a_f$= 700 μm)



# 2-D Stress Field in the Spring Critical Cross Section and the Location of the Initial Crack (*non-metallic inclusion*)

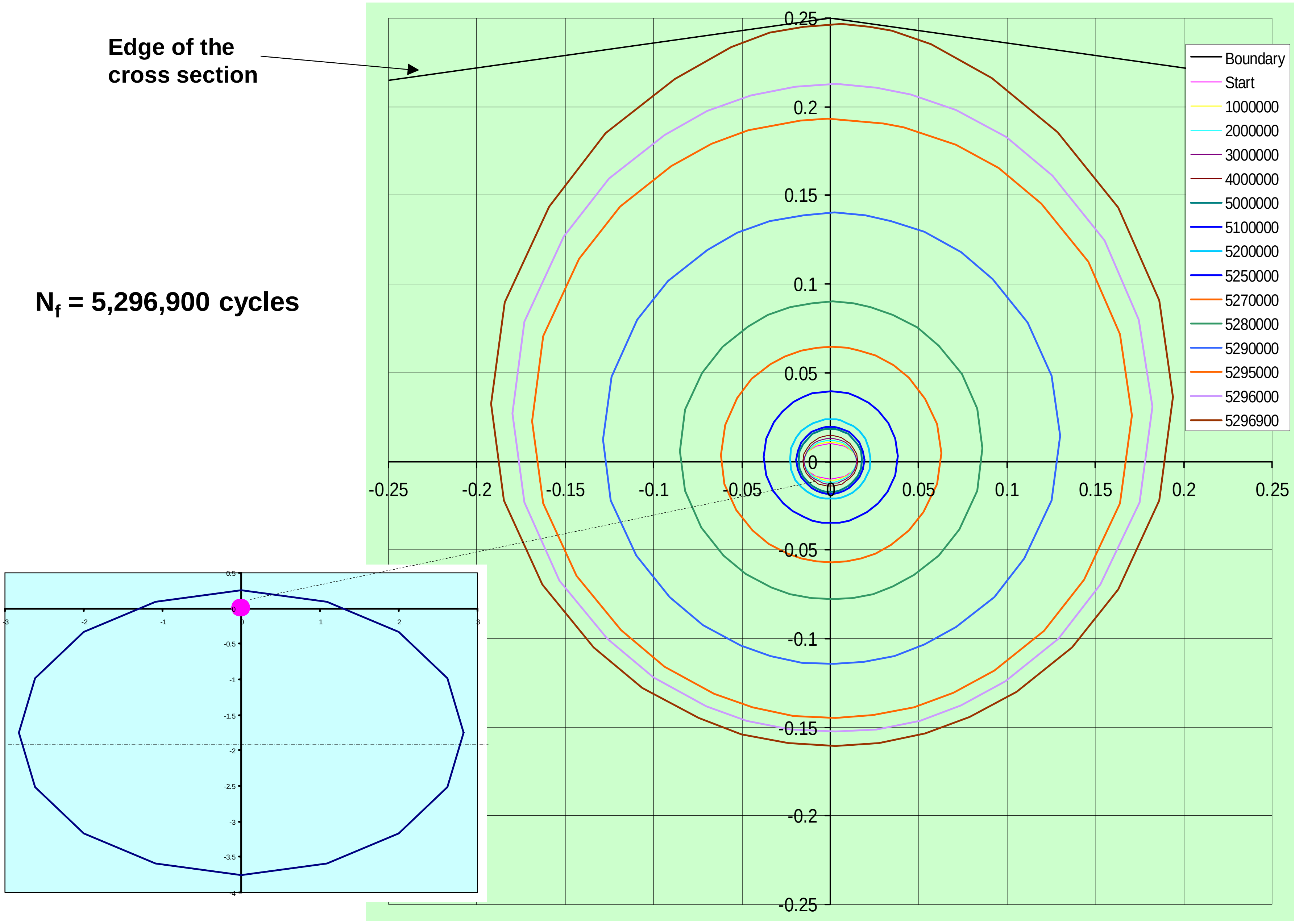
$$\sigma_{ref} = \sigma_{torque} = \frac{16T}{\pi d^3} = \frac{16F\frac{D}{2}}{\pi d^3} = \frac{8FD}{\pi d^3} = 4.625\frac{8F}{\pi d^2}$$



**Edge of the cross section**

**$N_f$ = 5,296,900 cycles**




Fatigue crack growth; d=0.03x0.02 mm, depth 0.25 mm, $\sigma_{C, max}$= 1030 MPa, $\sigma_{C, min}$= 390 MPa

# Main steps in fatigue design – flow chart